Chapter 4

Waves In Plasmas

4.1 Introduction

4.1.1 Plane waves

A wave may be defined as a propagating disturbance(?). If the disturbances are propagating in a single direction, the waves are called plane waves. For a monochromatic plane wave disturbance, a sinusoidally varying quantity in space and time is represented by

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{4.1}$$

where

- $\mathbf{A}_0$ is a constant vector defining the amplitude of the waves,
- $\mathbf{k}\cdot\mathbf{r} - \omega t$ is the phase,
- $\mathbf{k}$ is the wave vector,
- ω is the angular frequency.

The measurable quantity is understood to be the real part of the complex expression. If we allow $\mathbf{A}_0$ to be complex, we should replace as follows:

$$\mathbf{A}_0 \to \bar{\mathbf{A}}_0 e^{i\delta} = \bar{\mathbf{A}}_0 \cos\delta + i\bar{\mathbf{A}}_0 \sin\delta \tag{4.2}$$

$$\mathrm{Re}\,[\mathbf{A}(\mathbf{r}, t)] = \bar{\mathbf{A}}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta) \tag{4.3}$$

4.1.2 Phase velocity

The surface of a constant phase or the wave surface is displaced with a phase velocity,

$$\boxed{\mathbf{v}_p = \frac{d\mathbf{r}}{dt} = \frac{\omega}{k}\hat{\mathbf{k}}} \tag{4.4}$$

which is obtained from

$$\frac{d}{dt}(\mathbf{k}\cdot\mathbf{r} - \omega t) = 0\,. \tag{4.5}$$

The wavelength is the distance traveled by the wave when the phase has increased by 2π with t constant, so that

$$\boxed{\lambda = \frac{2\pi}{k}} \tag{4.6}$$

and the period is the time elapsed for a 2π phase shift at constant $\mathbf{r}$, or

$$\boxed{\tau = \frac{2\pi}{\omega}}. \tag{4.7}$$

The vector index of refraction $\mathbf{n}$ defined by

$$\boxed{\mathbf{n} = \frac{c}{\omega}\mathbf{k}} \tag{4.8}$$

can be used to simplify the form of the wave equation. Physically, it represents the ratio of the phase speed of the wave to the speed of light in free space.

4.1.3 Group velocity

A complex wave of arbitrary shape can be represented by a superposition of sinusoidal waves with different wave vectors and frequencies. Consider a wave packet made up of two waves,

$$A_1 = A\cos(k_1 x - \omega_1 t)$$

$$A_2 = A\cos(k_2 x - \omega_2 t)\,,$$

where $k_1 \simeq k_2 = k$ and $\omega_1 \simeq \omega_2 = \omega$. Using the identity

$$\cos a + \cos b = 2\cos\frac{1}{2}(a-b)\cos\frac{1}{2}(a+b)\,, \tag{4.9}$$

the resultant wave becomes

$$\begin{aligned} A_1 + A_2 &= 2A\cos\tfrac{1}{2}[(k_2 - k_1)x - (\omega_2 - \omega_1)t]\cos\tfrac{1}{2}[(k_2 + k_1)x - (\omega_2 + \omega_1)t] \\ &= 2A\cos\tfrac{1}{2}[(dk)x - (d\omega)t]\cos(kx - \omega t)\,. \end{aligned} \tag{4.10}$$

The second cosine function represents one of the orginal waves and the cosine function in front modulates this wave. The envelop of the resultant wave travels at velocity $d\omega/dk$, which is called the group velocity. In vetor form, the group velocity is

$$\boxed{\mathbf{v}_g = \frac{d\omega}{d\mathbf{k}}}. \tag{4.11}$$

More generally, a wave packet is

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{ikx - i\omega(k)t}dk \tag{4.12}$$

The amplitude $A(k)$ is given by the transform of the $u(x,0)$:

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0) e^{-ikx} dx \tag{4.13}$$

If $A(k)$ is fairly sharply peaked around some value k_0, then the frequency $\omega(k)$ can be expanded around that value of k:

$$\omega(k) = \omega_0 + \left.\frac{d\omega}{dk}\right|_0 (k - k_0) + \cdots \tag{4.14}$$

After performing the integral,

$$u(x,t) \simeq \frac{e^{i[k_0 (d\omega/dk)|_0 - \omega_0]t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{i[x - (d\omega/dk)|_0 t]k} dk \tag{4.15}$$

or

$$u(x,t) \simeq u\left(x - (d\omega/dk)|_0 t, 0\right) e^{i[k_0 (d\omega/dk)|_0 - \omega_0]t} \tag{4.16}$$

Therefore, apart from an overall phase factor, the pulse travels along un-distorted in shape with a velocity, which is the group velocity.

$$\boxed{v_g = \left.\frac{d\omega}{dk}\right|_0} \tag{4.17}$$

4.1.4 Type of Waves

- Longitudinal wave (e.g., sound wave, P-wave): $\mathbf{A} \parallel \mathbf{k} \quad (\mathbf{k} \times \mathbf{A} = 0)$
- Transverse wave (e.g., EM wave, S-wave): $\mathbf{A} \perp \mathbf{k} \quad (\mathbf{k} \cdot \mathbf{A} = 0)$
- Electrostatic wave: $\mathbf{E} \neq 0$, $\mathbf{B} = 0$ (no electrostatic wave in vacuum)
- Electromagnetic wave: $\mathbf{E} \neq 0$, $\mathbf{B} \neq 0$
- Parallel wave: $\mathbf{B}_0 \parallel \mathbf{k}$
- Perpendicular wave: $\mathbf{B}_0 \perp \mathbf{k}$

4.1.5 Polarization

Reference: **Optics**, Eugene Hecht (Shaum's outline series).

Consider two perpendicular electric fields given by

$$\begin{aligned} \mathbf{E}_x(z,t) &= \hat{x} E_{0x} \cos(kz - \omega t) \\ \mathbf{E}_y(z,t) &= \hat{y} E_{0y} \cos(kz - \omega t + \delta) \,. \end{aligned} \tag{4.18}$$

The waves move in the positive z-direction and have a relative phase δ. The resultant disturbance

$$\mathbf{E}(z,t) = \mathbf{E}_x(z,t) + \mathbf{E}_y(z,t) \tag{4.19}$$

varies with δ.

Linear polariztaion

When $\delta = 0$ or π,

$$\mathbf{E}(z,t) = (E_{0x}\hat{x} \pm E_{0y}\hat{y})\cos(kz - \omega t)\,. \tag{4.20}$$

The amplitude $E_{0x}\hat{x} \pm E_{0y}\hat{y}$ is constant vector and so the resultant wave is plane or linearly polarized.

Circular polarization

When $\delta = \pm\frac{\pi}{2}$ and $E_{0x} = E_{0y} \equiv E_0$,

$$\begin{aligned} \mathbf{E}_x(z,t) &= \ \hat{x}E_0\cos(kz - \omega t) \\ \mathbf{E}_y(z,t) &= \mp\hat{y}E_0\sin(kz - \omega t) \end{aligned} \tag{4.21}$$

and

$$\mathbf{E}(z,t) = E_0[\hat{x}\cos(kz - \omega t) \mp \hat{y}\sin(kz - \omega t)]\,. \tag{4.22}$$

The magnitude of $\mathbf{E}$ is E_0 is constant, but the direction of $\mathbf{E}$ is a function of z and t. When $\delta = -\frac{\pi}{2}$, the electric field rotates clockwise at a given point in space when viewed against the direction of propagation. Because the amplitude is constant, the endpoint of $\mathbf{E}$ sweeps out a circle with a frequency equal to that of the consitituent waves. Such a field is said to be right circular polarized. When $\delta = \frac{\pi}{2}$, $\mathbf{E}$ rotates counterclockwise and the field is left circular polarized.

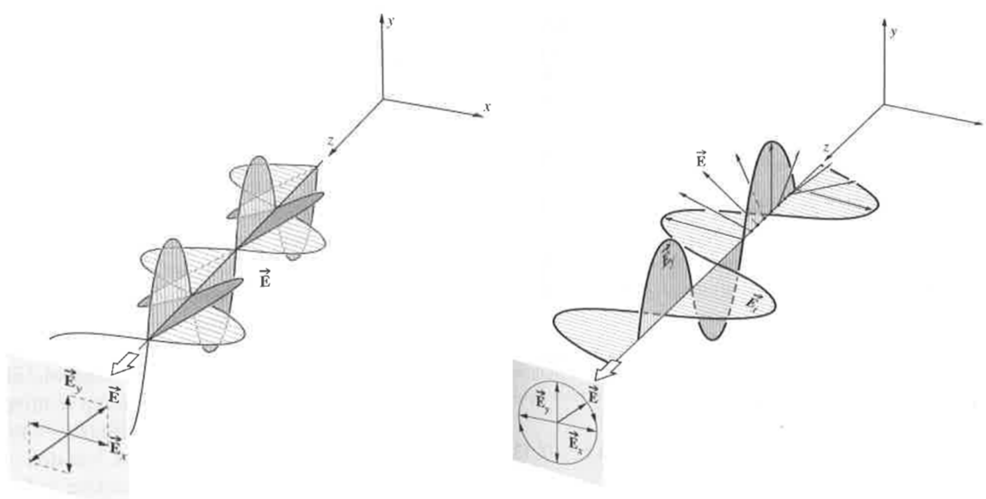

Figure 4.1: Linear polarization and circular polarization ($\delta = -\pi/2$, clockwise, right circular polarized, negative helicity, L-wave when $\mathbf{B} \parallel \hat{z}$). Here, fields are observed as a fixed point in space.

Elliptical polarization

For arbitrary phase difference and $E_{0x} \neq E_{0y}$, we obtain

$$\left(\frac{E_x}{E_{0x}}\right)^2 + \left(\frac{E_y}{E_{0y}}\right)^2 - 2\left(\frac{E_x}{E_{0x}}\right)\left(\frac{E_y}{E_{0y}}\right)\cos\delta = \sin^2\delta \tag{4.23}$$

which is the equation of an ellipse tilted at an angle ψ to the E_x axis, and θ is given by

$$\tan 2\psi = \frac{2E_{0x}E_{0y}}{E_{0x}^2 - E_{0y}^2}\cos\delta\,. \tag{4.24}$$

The tip of the field vector describes an ellipse and the wave is said to be elliptically polarized.

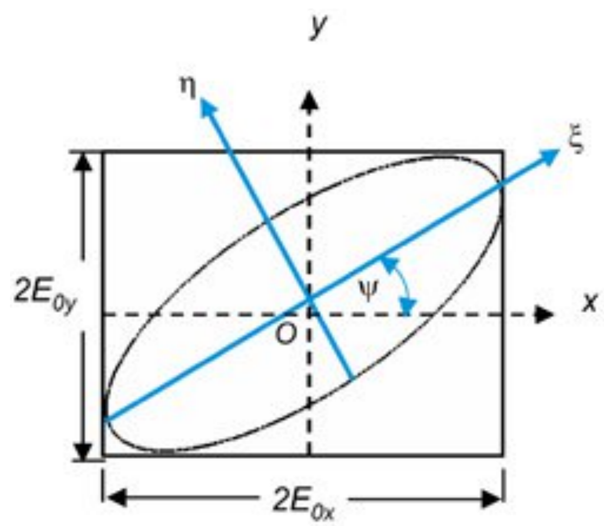

Figure 4.2:

4.2 Plasma Oscillation

If the electrons in a plasma are displaced from a uniform background ions,

1. **E** will be built up.
2. **E** will pull the electrons back to their original position.
3. The electrons will overshoot because of their inertia.
4. The electrons will oscillate (at the plasma frequency).

ASSUMPTIONS

- Homogeneous infinite plasma
- No external fields: $\mathbf{E}_0 = \mathbf{B}_0 = 0$
- Cold Plasma: $T_i = T_e = 0$

- Immobile ions: $\mathbf{v}_i = 0$, $n_i = n_0$

FLUID EQUATIONS

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}_e) = 0 \tag{4.25}$$

$$mn_e \left[\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \cdot \nabla)\mathbf{v}_e \right] = -n_e e \mathbf{E} \tag{4.26}$$

$$\nabla \cdot \mathbf{E} = \frac{e}{\epsilon_0}(n_i - n_e) \tag{4.27}$$

PERTURBATION EQUATIONS

Let

$$\begin{aligned} n_e(\mathbf{r},t) &= n_0 + && n_1(\mathbf{r},t) \\ \mathbf{v}_e(\mathbf{r},t) &= && \mathbf{v}_1(\mathbf{r},t) \\ \mathbf{E}(\mathbf{r},t) &= && \mathbf{E}_1(\mathbf{r},t) \end{aligned} \tag{4.28}$$

Then we have

$$\frac{\partial n_1}{\partial t} + \nabla \cdot (n_0 \mathbf{v}_1 + \underbrace{n_1 \mathbf{v}_1}_{neglect}) = 0 \tag{4.29}$$

$$m_e \left[\frac{\partial \mathbf{v}_1}{\partial t} + \underbrace{(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1}_{neglect} \right] = -e\mathbf{E}_1 \tag{4.30}$$

$$\nabla \cdot \mathbf{E}_1 = -\frac{e}{\epsilon_0} n_e \tag{4.31}$$

LINEARIZED EQUATIONS

$$\frac{\partial n_1}{\partial t} + n_0 \nabla \cdot \mathbf{v}_1 = 0 \tag{4.32}$$

$$m_e \frac{\partial \mathbf{v}_1}{\partial t} = -e\mathbf{E}_1 \tag{4.33}$$

$$\nabla \cdot \mathbf{E}_1 = -\frac{e}{\epsilon_0} n_1 \tag{4.34}$$

Assume that

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{v}_1 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \\ n_1 &= n_1 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \\ \mathbf{E}_1 &= \mathbf{E}_1 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \end{aligned} \tag{4.35}$$

Then the linearized perturbation equations become

$$-i\omega n_1 + n_0 i\mathbf{k}\cdot\mathbf{v}_1 = 0 \tag{4.36}$$

$$-m_e i\omega\mathbf{v}_1 = -e\mathbf{E}_1 \tag{4.37}$$

$$i\mathbf{k}\cdot\mathbf{E}_1 = -\frac{e}{\epsilon_0}n_1\,. \tag{4.38}$$

Note that

- From $\nabla\times\mathbf{E}_1 = 0$, $\mathbf{k}\times\mathbf{E}_1 = 0$ ($\mathbf{E}_1$ is longitudinal).
- From the momentum equation Eq. (4.33), $\mathbf{v}_1 \parallel \mathbf{E}_1$.

Since $\mathbf{k}\parallel\mathbf{v}_1\parallel\mathbf{E}_1$,

$$-\omega n_1 + n_0 k v_1 = 0 \tag{4.39}$$

$$-m_e i\omega v_1 + eE_1 = 0 \tag{4.40}$$

$$\frac{e}{\epsilon_0}n_1 + ikE_1 = 0 \tag{4.41}$$

This is a set of 3 equations with 3 unknowns.

DISPERSION RELATION

The condition for nontrivial solutions is

$$\begin{vmatrix} -\omega & n_0 k & 0 \\ 0 & -im_e\omega & e \\ \frac{e}{\epsilon_0} & 0 & ik \end{vmatrix} = 0 \tag{4.42}$$

$$-m_e\omega^2 k + \frac{n_0}{\epsilon_0}e^2 k = 0\,. \tag{4.43}$$

Or

$$\omega = \omega_{pe}$$

where

$$\boxed{\omega_{pe} = \sqrt{\frac{n_0 e^2}{m_e\epsilon_0}}} \quad \text{(electron) plasma frequency} \tag{4.44}$$

NOTES

- $\omega = \omega_{pe}$ (=constant): The disturbance oscillates at the plasma frequency ω_{pe} Plasma oscillations are also called Langmuir oscillations.
- $v_g = \frac{d\omega}{dk} = 0$: The disturbance does not propagates. Motions of charged particles are local oscillations confined to the region where the perturbaions occur.

- These oscillations involve an interchange of the energy between the electric field and the kinetic energy of particle motion. It can be shown that there is no magnetic field energy associated with the oscillation, since the convection current density and displacement current density associated with these oscillations cancel each other (there is no net current in the plasma).
- The plasma approximation is not applicable. In plasma oscillations, electron inertia is important (high frequency motion).

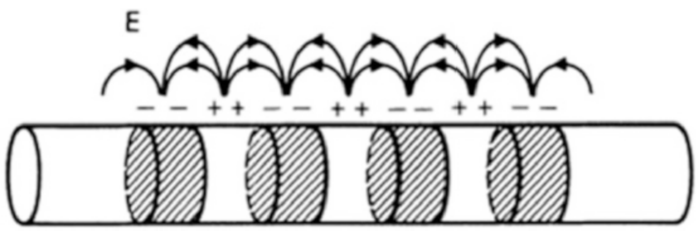

Figure 4.3: In infinite plane system, the electric field due to equal numbers of positive and negative charge sheets is zero. In any finite system, however, plasma oscillations will propagate. The fringing electric field causes a coupling of the disturbance to adjacent layers, and the oscillation does not stay localized.

4.3 Electron Plasma Waves

ASSUMPTIONS

- Homogenious quasineutral plasma
- No external field: $\mathbf{E}_0 = \mathbf{B}_0 = 0$
- Cold ions and warm elections: $T_i = 0$, but $\underline{T_e \neq 0} \longrightarrow \nabla p_e = \gamma_e K T_e \nabla n$.
- Immobile ions: $\mathbf{v}_i = 0$, $n_i = n_0$

FLUID EQUATIONS

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}_e) = 0 \tag{4.45}$$

$$m_e n_e \left[\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \cdot \nabla)\mathbf{v}_e \right] = -n_e e \mathbf{E} - \gamma_e K T_e \nabla n_e \tag{4.46}$$

$$\nabla \cdot \mathbf{E} = \frac{e}{\epsilon_0}(n_i - n_e) \tag{4.47}$$

LINEARIZED EQUATIONS

$$\frac{\partial n_1}{\partial t} + n_0 \nabla \cdot \mathbf{v}_1 = 0 \tag{4.48}$$

$$mn_0 \frac{\partial \mathbf{v}_1}{\partial t} = -en_0 \mathbf{E}_1 - \gamma_e KT_e \nabla n_1 \tag{4.49}$$

$$\nabla \cdot \mathbf{E}_1 = -\frac{e}{\epsilon_0} n_1 \tag{4.50}$$

The linearized perturbation equations become

$$-i\omega n_1 + n_0 i\mathbf{k} \cdot \mathbf{v}_1 = 0 \tag{4.51}$$

$$-m_e n_0 i\omega \mathbf{v}_1 = -en_0 \mathbf{E}_1 - i\gamma_e KT_e n_1 \mathbf{k} \tag{4.52}$$

$$i\mathbf{k} \cdot \mathbf{E}_1 = -\frac{e}{\epsilon_0} n_1 \tag{4.53}$$

Since $\mathbf{k} \parallel \mathbf{v}_1 \parallel \mathbf{E}_1$,

$$-\omega n_1 + n_0 k v_1 = 0 \tag{4.54}$$

$$i\gamma_e KT_e k n_1 - m_e n_0 i\omega v_1 + n_0 e E_1 = 0 \tag{4.55}$$

$$\frac{e}{\epsilon_0} n_1 + ikE_1 = 0 \tag{4.56}$$

DISPERSION RELATION

$$\begin{vmatrix} -\omega & n_0 k & 0 \\ ik\gamma_e KT_e & -im_e n_0 \omega & n_0 e \\ \frac{e}{\epsilon_0} & 0 & ik \end{vmatrix} = 0 \tag{4.57}$$

$$-m_e n_0 \omega^2 k + \frac{n_0}{\epsilon_0} e^2 k + \gamma_e n_0 KT_e k^3 = 0 \tag{4.58}$$

Or

$$\omega^2 = \omega_p^2 + \left(\frac{\gamma_e KT_e}{m_e} \right) k^2 \tag{4.59}$$

Since $\lambda_D^2 = \frac{\epsilon_0 KT}{ne^2}$,

$$\boxed{\omega^2 = \omega_p^2 \left(1 + \gamma_e k^2 \lambda_D^2 \right)}. \tag{4.60}$$

Consider long wave length waves, such that a typical electrons travels only a fraction of a wavelength λ in one wave period; then the compression of the wave will be an adiabatic one. Thus, the assumption $v_e/\omega \ll \lambda$, or

$$\frac{\omega}{k} \gg v_e \tag{4.61}$$

leads us to consider adiabatic compression. Without collisions the change in temperature during the compression along the direction of wave propagating will not be transmitted to the other two directions, so that the compression is a one dimensional adiabatic process or $\gamma_e = 3$.

$$\frac{\gamma_e KT_e}{m} = \frac{3KT_e}{m_e} = \frac{3}{2} v_{the}^2 \tag{4.62}$$

So

$$\boxed{\omega^2 = \omega_p^2 + \frac{3}{2}k^2 v_e^2} \tag{4.63}$$

NOTES

- Phase velocity:

$$v_\phi = \frac{\omega}{k} = \frac{\sqrt{\frac{3}{2}}v_e}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} > \sqrt{\frac{3}{2}}v_e$$

- Group velocity:

$$v_g = \frac{d\omega}{dk} = \frac{k}{\omega}\frac{3}{2}v_e^2 = \sqrt{\frac{3}{2}}v_e\sqrt{1 - \frac{\omega_p^2}{\omega^2}} < \sqrt{\frac{3}{2}}v_e$$

- Electron plasma waves (Langmuir waves) are one of electrostatic waves, which is generated by when a perturbation creats a charge imbalance in a neutral fluid element. The charge imbalance accelerates electrons in the neighborhood of the charged fluid element, resulting in charge oscillating back and forth. These oscillations involve only the electric field and they are defined as electrostatic waves (when oscillations propagates).

- The relation $\omega = \omega(k)$ is called the dispersion relation. Wave components with different k travel with different speeds so that a wave profile, which can be represented by the sum of wave components, will necessarily spread out or *disperse*. Either ω or k can be viewed as the independent variable when one considers making a linear superposition.

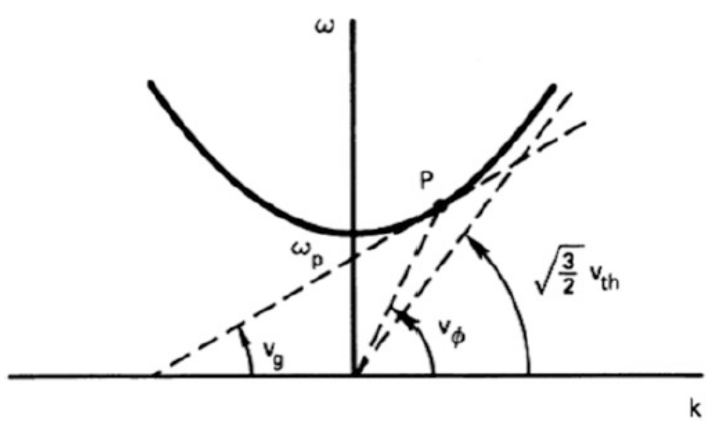

Figure 4.4:

4.4 Sound Waves

FLUID EQUATIONS

For a neutral gas, we have

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0 \tag{4.64}$$

$$mn\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right] = -\nabla p \tag{4.65}$$

$$\nabla p = \gamma KT \nabla n \tag{4.66}$$

LINEAERIZED EQUATIONS

$$\frac{\partial n_1}{\partial t} + n_0 \nabla \cdot \mathbf{v}_1 = 0 \tag{4.67}$$

$$mn_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p = -\gamma KT \nabla n \tag{4.68}$$

The linearized perturbation equations become

$$-i\omega n_1 + n_0 i\mathbf{k} \cdot \mathbf{v}_1 = 0 \tag{4.69}$$

$$-i\omega m n_0 \mathbf{v}_1 = -i\mathbf{k}\gamma KT n_1 \tag{4.70}$$

or

$$\omega n_1 - n_0 k v_1 = 0 \tag{4.71}$$

$$k\gamma KT n_1 - \omega m n_0 v_1 = 0 \tag{4.72}$$

DISPERSION RELATION

To have non-trivial solutions, we require

$$\begin{vmatrix} \omega & -n_0 k \\ k\gamma KT & -\omega m n_0 \end{vmatrix} = 0$$

Or

$$\frac{\omega^2}{k^2} = \frac{\gamma KT}{m} \tag{4.73}$$

Hence,

$$\omega^2 = c_s^2 k^2 \tag{4.74}$$

where

$$\boxed{c_s = \sqrt{\frac{\gamma KT}{m}}} \quad \text{sound speed} \tag{4.75}$$

NOTES

- Phase velocity: $v_\phi = \frac{\omega}{k} = c_s$.
- Group velocity: $v_g = \frac{d\omega}{dk} = c_s$.
- $\mathbf{v}_1 \parallel \mathbf{k}$: longitudinal.
  Disturbances are propagating by collisions among molecules.