4.13 Hydromagnetic Waves

4.13.1 Alfvén wave

ASSUMPTIONS

- Homogeneous infinite quasineutral plasma
- External field: $\mathbf{E}_0 = 0$, $\mathbf{B}_0 \neq 0$ ($\mathbf{B}_0 \parallel \mathbf{k}$)
- Cold Plasma: $T_i = T_e = 0$
- Moving ions: $\mathbf{v}_{i1} \neq 0$, $n_{i1} \neq 0$
- $\omega \ll \omega_{ci}$

FLUID EQUATIONS

$$m_e n_e \left[\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \cdot \nabla)\mathbf{v}_e\right] = -n_e e\mathbf{E} - e n_e \mathbf{v}_e \times \mathbf{B} \tag{4.202}$$

$$m_i n_i \left[\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \cdot \nabla)\mathbf{v}_i\right] = +n_i e\mathbf{E} + e n_i \mathbf{v}_i \times \mathbf{B} \tag{4.203}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.204}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{4.205}$$

with

$$\mathbf{J} = e n_0 (\mathbf{v}_i - \mathbf{v}_e) \tag{4.206}$$

LINEARIZED EQUATIONS

Let $\mathbf{E}_1 = E_1 \hat{x}$ and $\mathbf{k} = k\hat{z}$.
The momentum equation for electrons is

$$m_e \frac{\partial \mathbf{v}_{e1}}{\partial t} = -e\mathbf{E}_1 - e\mathbf{v}_{e1} \times \mathbf{B}_0 \tag{4.207}$$

$$\longrightarrow \begin{cases} i\omega v_{ex} = \dfrac{e}{m_e} E_x + \omega_{ce} v_{ey} \\ i\omega v_{ey} = \dfrac{e}{m_e} E_y - \omega_{ce} v_{ex} \\ i\omega v_{ez} = 0 \end{cases} \tag{4.208}$$

$$\longrightarrow \begin{cases} v_{ex} = \dfrac{e}{m_e\omega}(-iE_1)\left(1 - \dfrac{\omega_{ce}^2}{\omega^2}\right)^{-1} \simeq \dfrac{ie}{m_e\omega}\dfrac{\omega^2}{\omega_{ce}^2}E_1 \\ v_{ey} = \dfrac{e}{m_e\omega}\dfrac{\omega_{ce}}{\omega}E_1\left(1 - \dfrac{\omega_{ce}^2}{\omega^2}\right)^{-1} \simeq -\dfrac{e}{m_e\omega}\dfrac{\omega_{ce}}{\omega}\dfrac{\omega^2}{\omega_{ce}^2}E_1 = -\dfrac{E_1}{B_0} : \quad \mathbf{E}_1 \times \mathbf{B}_0 \text{ drift} \\ v_{ez} = 0 \end{cases} \tag{4.209}$$

The momentum equation for ions is

$$m_i\frac{\partial \mathbf{v}_{i1}}{\partial t} = +e\mathbf{E}_1 + e\mathbf{v}_{i1} \times \mathbf{B}_0 \tag{4.210}$$

$$\longrightarrow \begin{cases} i\omega v_{ix} = -\dfrac{e}{m_i}E_x - \omega_{ci}v_{iy} \\ i\omega v_{iy} = -\dfrac{e}{m_i}E_y + \omega_{ci}v_{ix} \\ i\omega v_{iz} = 0 \end{cases} \tag{4.211}$$

$$\longrightarrow \begin{cases} v_{ix} = \dfrac{e}{m_i\omega}(iE_1)\left(1 - \dfrac{\omega_{ci}^2}{\omega^2}\right)^{-1} \simeq \dfrac{-ieE_1}{m_i\omega}\dfrac{\omega^2}{\omega_{ci}^2} \\ v_{iy} = \dfrac{e}{m_i\omega}\dfrac{\omega_{ci}}{\omega}E_1\left(1 - \dfrac{\omega_{ci}^2}{\omega^2}\right)^{-1} \simeq \dfrac{eE_1}{m_i\omega_{ci}} = -\dfrac{E_1}{B_0} : \quad \mathbf{E}_1 \times \mathbf{B}_0 \text{ drift} \\ v_{iz} = 0 \end{cases} \tag{4.212}$$

The wave equation is

$$\nabla(\nabla \cdot \mathbf{E}_1) - \nabla^2\mathbf{E}_1 = -\mu_0\frac{\partial \mathbf{J}_1}{\partial t} - \mu_0\epsilon_0\frac{\partial^2\mathbf{E}_1}{\partial t^2} \tag{4.213}$$

$$\longrightarrow -\mathbf{k}(\mathbf{k} \cdot \mathbf{E}_1) + k^2\mathbf{E}_1 = \mu_0 e n_0 i\omega(\mathbf{v}_{i1} - \mathbf{v}_{e1}) + \frac{\omega^2}{c^2}\mathbf{E}_1 \tag{4.214}$$

$$\longrightarrow (\omega^2 - k^2c^2)\mathbf{E}_1 = -\frac{i\omega e n_0}{\epsilon_0}(\mathbf{v}_{i1} - \mathbf{v}_{e1}) \tag{4.215}$$

since

$$\mathbf{J} = en_0(\mathbf{v}_{i1} - \mathbf{v}_{e1}) \tag{4.216}$$

DISPERSION RELATION

$$(\omega^2 - k^2c^2)\mathbf{E}_1 = -\frac{ien_0}{\epsilon_0}ie\left(-\frac{\omega^2}{m_i\omega_{ci}^2} - \frac{\omega^2}{m_e\omega_{ce}^2}\right)\mathbf{E}_1 \simeq -\omega^2\frac{n_0 m_i}{\epsilon_0 B_0^2}\mathbf{E}_1 \tag{4.217}$$

$$(\omega^2 - k^2c^2)\mathbf{E}_1 = -\omega^2\frac{\rho_m}{\epsilon_0 B_0^2}\mathbf{E}_1 \tag{4.218}$$

For $E_1 \neq 0$,

$$(\omega^2 - k^2c^2) = -\omega^2\frac{\rho_m}{\epsilon_0 B_0^2} \tag{4.219}$$

where ρ_m is the mass density $n_o m_i$. Thus we obtain the dispersion relation

$$\frac{\omega^2}{k^2} = \frac{c^2}{1+\frac{\rho_m}{\epsilon_0 B_0^2}} = \frac{1}{\mu_0\epsilon_0\left(1+\frac{\rho_m}{\epsilon_0 B_0^2}\right)} \simeq \frac{B_0^2}{\mu_0\rho_m} \tag{4.220}$$

$$v_\phi = \frac{\omega}{k} = \frac{B_0}{\sqrt{\mu_0\rho_m}} \equiv v_A : \quad \text{Alfvén velocity} \tag{4.221}$$

NOTES

- Physical interpretation:

  1. Electrons and ions are drifting together in the y-direction ($\mathbf{E}\times\mathbf{B}$ drift), with speed $-E_1/B_0$.
     Thus both plasma fluids move together in the y-direction.
  2. Magnetic field lines are distorted by the addition of $\mathbf{B} = B_1\hat{y}$ to the external magetic field $\mathbf{B}_0 = B_0\hat{z}$.
     Since $i\mathbf{k}\times\mathbf{E}_1 = +i\omega\mathbf{B}_1$,

     $$B_1 = +\frac{k}{\omega}E_1 \tag{4.222}$$

     The $B_1\hat{y}$ propagates with speed $\frac{\omega}{k}$ in the z-direction since $\mathbf{k} = k\hat{z}$.
     The velocity of the field line in the y-direction is

     $$-\frac{B_1}{B_0}\frac{\omega}{k} = -\frac{E_1}{B_0} \tag{4.223}$$

     which is precisely the y-direction of fluid flow.

  3. The field lines and the particles oscillate together as if they were stuck together.

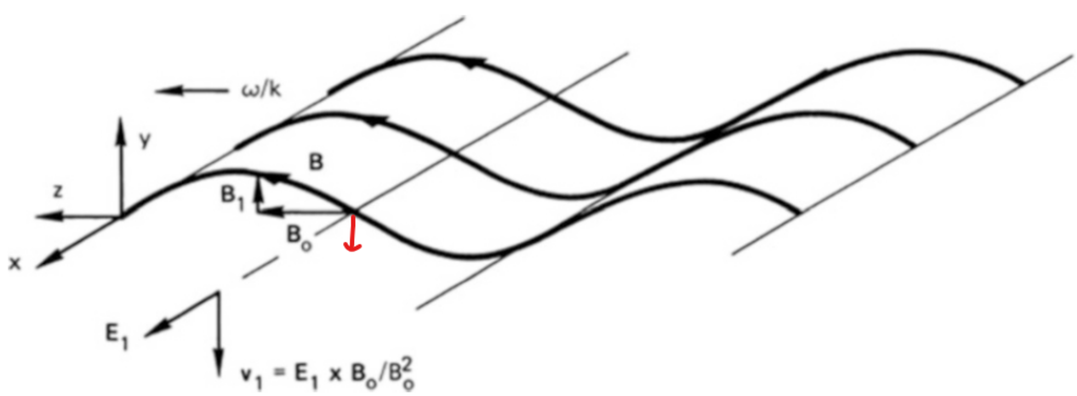

Figure 4.12: Relation among the oscillating quantities in an Alfvén wave and the (exaggerated) distortion of the lines of force.

- From classical physics

$$v = \sqrt{\frac{T}{\rho_l}} \tag{4.224}$$

where v is the velocity of propagation, T is the tension of the string, and ρ_l is the mass per unit length.
For Alfvén waves

$$T \rightarrow \quad \text{tension/unit area of magnetic field} \quad = \frac{B_0^2}{\mu_0}$$

Each particle is constrained to circle a particular B-line. Thus the B-line has a mass associated with it–the mass of the particle tied to it. Thus $\rho_l \rightarrow \rho_m$.

$$v = \sqrt{\frac{B_0^2}{\mu_0 \rho_m}} \equiv v_A \tag{4.225}$$

4.13.2 Magnetosonic Wave

ASSUMPTIONS

- Homogeneous infinite quasineutral plasma
- External field: $\mathbf{E}_0 = 0$, $\mathbf{B}_0 \neq 0$ ($\mathbf{B}_0 \perp \mathbf{k}$)
- Hot Plasma: $T_i \neq 0$ $T_e \neq 0$
- Moving ions: $\mathbf{v}_{i1} \neq 0$, $n_{i1} \neq 0$
- $\omega \ll \omega_{ci}$

FLUID EQUATIONS

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{v}_i) = 0 \tag{4.226}$$

$$m_i n_i \left[\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \cdot \nabla)\mathbf{v}_i \right] = +n_i e\mathbf{E} + en_i \mathbf{v}_i \times \mathbf{B} - \gamma_i K T_i \nabla n_i \tag{4.227}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}_e) = 0 \tag{4.228}$$

$$m_e n_e \left[\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \cdot \nabla)\mathbf{v}_e \right] = -n_e e\mathbf{E} - en_e \mathbf{v}_e \times \mathbf{B} - \gamma_e K T_e \nabla n_e \tag{4.229}$$

$$\mathbf{J} = en_0(\mathbf{v}_i - \mathbf{v}_e) \tag{4.230}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.231}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{4.232}$$

LINEARIZED EQUATIONS

Let $\mathbf{E}_1 = E_1\hat{x}$ and $\mathbf{k} = k\hat{y}$.
The wave equation is

$$\nabla(\nabla \cdot \mathbf{E}_1) - \nabla^2 \mathbf{E}_1 = -\mu_0 \frac{\partial \mathbf{J}_1}{\partial t} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}_1}{\partial t^2} \tag{4.233}$$

$$\longrightarrow -\mathbf{k}(\mathbf{k} \cdot \mathbf{E}_1) + k^2 \mathbf{E}_1 = \mu_0 e n_0 i\omega(\mathbf{v}_{i1} - \mathbf{v}_{e1}) + \frac{\omega^2}{c^2}\mathbf{E}_1 \tag{4.234}$$

$$\longrightarrow (\omega^2 - k^2c^2)\mathbf{E}_1 = -\frac{i\omega e n_0}{\epsilon_0}(\mathbf{v}_{i1} - \mathbf{v}_{e1}) \tag{4.235}$$

Since $\mathbf{E}_1 = E_1\hat{x}$,

$$(\omega^2 - k^2c^2)E_1 = -\frac{i\omega e n_0}{\epsilon_0}(v_{ix} - v_{ex}) \tag{4.236}$$

so that we need only x-components of velocities.
From the ion continuity equation

$$-i\omega n_{i1} + n_0 i\mathbf{k} \cdot \mathbf{v}_{i1} = 0 \longrightarrow \frac{n_{i1}}{n_0} = \frac{k}{\omega} v_{iy} \tag{4.237}$$

The ion momentum equation reads

$$m_i \frac{\partial \mathbf{v}_{i1}}{\partial t} = e(\mathbf{E}_1 + \mathbf{v}_{i1} \times \mathbf{B}_0) - \gamma_i K T_i \nabla n_i / n_0 \tag{4.238}$$

$$\longrightarrow \begin{cases} -i\omega v_{ix} = \dfrac{eE_1}{m_i} + \omega_{ci} v_{iy} \\ -i\omega v_{iy} = -\omega_{ci} v_{ix} - \dfrac{ik\gamma_i K T_i}{m_i} \dfrac{n_1}{n_0} \end{cases} \tag{4.239}$$

$$\longrightarrow \begin{cases} -i\omega v_{ix} - \omega_{ci} v_{iy} = \dfrac{eE_1}{m_i} \\ \omega_{ci} v_{ix} - i\omega \left[1 - \dfrac{k^2}{\omega^2}\dfrac{\gamma_i K T_i}{m_i}\right] v_{iy} = 0 \end{cases} \tag{4.240}$$

Thus we obtain

$$\left[1 - \frac{\omega_{ci}^2}{\omega^2}\left(1 - \frac{k^2}{\omega^2}\frac{\gamma_i K T_i}{m_i}\right)^{-1}\right] v_{ix} = \frac{ieE_1}{m_i \omega} \tag{4.241}$$

Since $\omega \ll \omega_{ci}$,

$$-\frac{\omega_{ci}^2}{\omega^2}\frac{1}{1 - A} v_{ix} = \frac{ieE_1}{m_i \omega} \tag{4.242}$$

where

$$A = \frac{k^2}{\omega^2}\frac{\gamma_i K T_i}{m_i} \tag{4.243}$$

Finally,

$$v_{ix} = \frac{ieE_1}{m_i\omega}\frac{\omega^2}{\omega_{ci}^2}(A-1) \quad (4.244)$$

For electrons,

$$\left[1 - \frac{\omega_{ce}^2}{\omega^2}\left(1 - \frac{k^2}{\omega^2}\frac{\gamma_e KT_e}{m_e}\right)^{-1}\right] v_{ex} = \frac{-ieE_1}{m_e\omega} \quad (4.245)$$

Since $\omega \ll \omega_{ce}$,

$$\left[-\frac{\omega_{ce}^2}{\omega^2}\left(1 - \frac{k^2}{\omega^2}\frac{\gamma_e KT_e}{m_e}\right)^{-1}\right] v_{ex} = \frac{-ieE_1}{m_e\omega} \quad (4.246)$$

Taking the limit $m_e \longrightarrow 0$ ($k^2 v_{the}^2/\omega^2 \gg 1$),

$$v_{ex} = \frac{-ieEk^2}{\omega\omega_{ce}^2}\frac{\gamma_e KT_e}{m_e^2} \quad (4.247)$$

DISPERSION RELATION

The wave equation becomes

$$(\omega^2 - c^2k^2)E_1 = \frac{e^2 n_0}{m_i\epsilon_0}\frac{\omega^2}{\omega_{ci}^2}(A-1)E_1 + \frac{e^2 n_0 k^2 \gamma_e KT_e}{\epsilon_0 m_e^2 \omega_{ce}^2}E_1 \quad (4.248)$$

For $E_1 \neq 0$, we require that

$$\omega^2 - c^2k^2 = \omega_{pi}^2\frac{\omega^2}{\omega_{ci}^2}(\frac{k^2}{\omega^2}\frac{\gamma_i KT_i}{m_i} - 1) + \frac{c^2k^2}{v_A^2}\frac{\gamma_e kT_e}{m_i} \quad (4.249)$$

Since

$$\frac{\omega_{pi}^2}{\omega_{ci}^2} = \frac{n_0 e^2}{m_i\epsilon_0}\frac{m_i^2}{e^2B_0^2} = \frac{n_0 m_i}{\epsilon_0 B_0^2} = \frac{c^2}{v_A^2} \quad (4.250)$$

$$\omega^2\left(1 + \frac{c^2}{v_A^2}\right) = k^2c^2\left(1 + \frac{\gamma_e KT_e + \gamma_i KT_i}{m_i v_A^2}\right) = k^2c^2\left(\frac{v_A^2 + v_s^2}{v_A^2}\right) \quad (4.251)$$

Or

$$\frac{\omega^2}{k^2} = c^2\left(\frac{v_s^2 + v_A^2}{c^2 + v_A^2}\right) \quad (4.252)$$

NOTES

- The magnetosonic wave is an acoustic wave in which the compressions and rarefactions are produced by $\mathbf{E} \times \mathbf{B}$ drift.
- $$\frac{\omega^2}{k^2} = c^2\left(\frac{v_s^2 + v_A^2}{c^2 + v_A^2}\right) \simeq v_s^2 + v_A^2 \quad (4.253)$$

  The phase velocity of the magnetosonic wave is almost always larger than v_A: it is called the fast hydromagnetic wave.
- In the limit $\mathbf{B}_0 \longrightarrow 0$, $v_A \longrightarrow 0$ so that $\frac{\omega^2}{k^2} \longrightarrow v_s^2$: ion acoustic wave.
- In the limit $T \longrightarrow 0$, $v_s \longrightarrow 0$ so that $\frac{\omega^2}{k^2} \longrightarrow v_A^2$: Alfvén wave

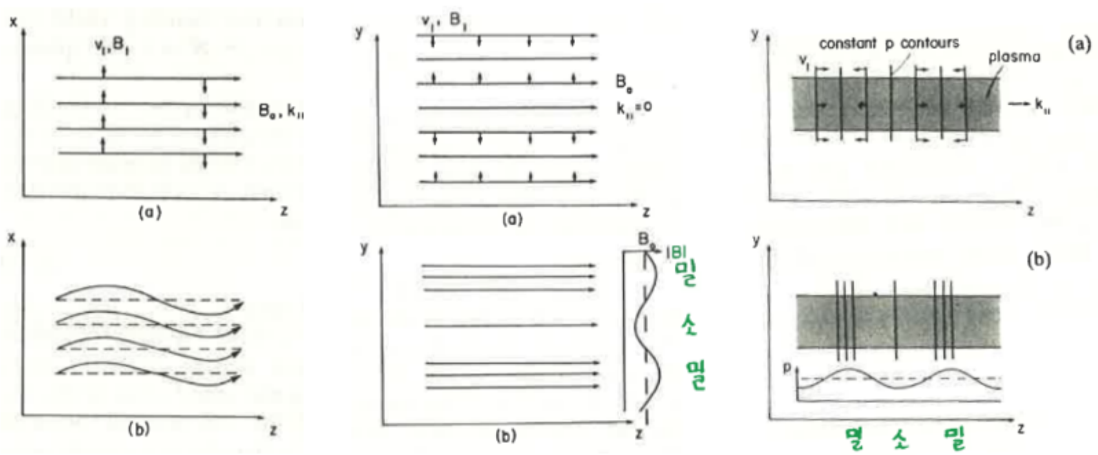

Figure 4.13: Magnetic and velocity perturbations for the shear (torsional, incompressible) Alfvén wave [left (a) and (b)]. Magnetic and velocity perturbations for the compressional Alfvén wave (or fast magnetosonic wave in the low$-\beta$ limit) [middle (a) and (b)]. Velocity and pressure perturbations for the sound wave (or slow magnetosonic wave in the low$-\beta$ limit) [right (a) and (b)].