Chapter 5

Diffusion and Resistivity

5.1 Diffusion and Mobility in Weakly Ionized Gases

5.1.1 Collisional Parameters

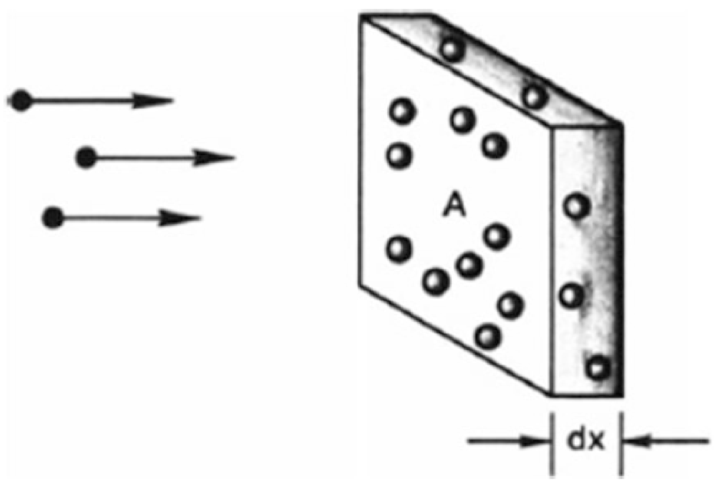

Figure 5.1:

The number of atoms in the slab:

$$nAdx$$

The fraction of the slab blocked by atoms:

$$\frac{\sigma}{A}nAdx = n\sigma dx$$

The flux out of the slab Γ' for the incident flux Γ:

$$\Gamma' = \Gamma(1 - n\sigma dx)$$

$$\frac{d\Gamma}{dx} = -n\sigma\Gamma$$

$$\Gamma = \Gamma_0 e^{-n\sigma x} \quad (5.1)$$

This represents the statistical average for a large number of particles and scatterers. This can be also interpreted as the probability that any given particle will penetrate a distance x into a gas.
The mean free path,λ_m, is defined as the average distance that a particle travels before colliding with a gas atom.

$$\lambda_m = \langle x \rangle = \frac{\int_0^\infty x e^{-n\sigma x} dx}{\int_0^\infty e^{-n\sigma x} dx} = \frac{1}{n\sigma} : \quad \text{mean free path} \tag{5.2}$$

$$\tau = \frac{\lambda_m}{v} : \quad \text{mean time between collisons}$$

$$\frac{1}{\tau} = \frac{v}{\lambda_m} = n\sigma v : \quad \text{mean frequency of collisions} \tag{5.3}$$

$$\nu = n \langle \sigma v \rangle : \quad \text{collision frequency} \tag{5.4}$$

5.1.2 Diffusion Parameters

The fluid equation of motion ($\mathbf{B} = 0$):

$$mn \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right] = \pm en\mathbf{E} - \nabla p - mn\nu\mathbf{v} \tag{5.5}$$

Assume

1. A steady state, $\frac{\partial}{\partial t} \longrightarrow 0$
2. Sufficiently small v or sufficiently large ν, $\mathbf{v} \cdot \nabla \longrightarrow 0$

Then for an isothermal plasma

$$\begin{aligned} \mathbf{v} &= \frac{1}{mn\nu}(\pm en\mathbf{E} - KT\nabla n) \\ &= \pm \frac{e}{m\nu}\mathbf{E} - \frac{KT}{m\nu}\frac{\nabla n}{n} \\ &= \pm \mu \mathbf{E} - D\frac{\nabla n}{n} \end{aligned} \tag{5.6}$$

where

$$\boxed{\mu = \frac{|q|}{m\nu}} \quad \text{Mobility} \tag{5.7}$$

$$\boxed{D = \frac{KT}{m\nu}} \quad \text{Diffusion Coefficient} \tag{5.8}$$

Note that the dimension of the diffusion coefficient is $[L]^2/[T]$.
Einstein Relation:

$$\mu = \frac{|q|}{KT} D \tag{5.9}$$

The flux of the jth species is defined by

$$\mathbf{\Gamma}_j = n\mathbf{v}_j = \pm \mu_j n\mathbf{E} - D_j \nabla n_j \tag{5.10}$$

Then equation of continuity is given as

$$\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{\Gamma} = 0 \tag{5.11}$$

If $\mathbf{E} = 0$ or $q = 0$,

$$\boxed{\mathbf{\Gamma}_j = -D_j \nabla n_j} \quad \text{Fick's Law} \tag{5.12}$$

5.2 Decay of a Plasma by diffusion

5.2.1 Ambipolar Diffusion

In the presence of a gradient in plasma density, both the electrons and ions will tend to diffuse into the region of lower density.

1. The electrons tends to diffuse more rapidly than the ions, due to their lighter mass.
2. There will be a space charge separation.
3. The resulting electric field will retard the electron diffusion and increase the ion diffusion so that space charge neutrality is maintained at all points in space.
4. Under these conditions, the electrons and ions will diffuse at the same rate as determined by the ambipolar diffusion coefficients.

For ions,

$$\mathbf{\Gamma}_i = -D_i \nabla n_i + \mu_i n_i \mathbf{E} \tag{5.13}$$

For electrons,

$$\mathbf{\Gamma}_e = -D_e \nabla n_e - \mu_e n_e \mathbf{E} \tag{5.14}$$

From the continuity equation, we have

$$\frac{\partial n_i}{\partial t} = -\nabla \cdot \mathbf{\Gamma}_i = D_i \nabla^2 n_i - \mu_i \nabla \cdot (n_i \mathbf{E}) \tag{5.15}$$

$$\frac{\partial n_e}{\partial t} = -\nabla \cdot \mathbf{\Gamma}_e = D_e \nabla^2 n_i + \mu_e \nabla \cdot (n_e \mathbf{E}) \tag{5.16}$$

From the space charge neutrality, $n_i \simeq n_e \equiv n$.
Multiplying the first equation by μ_e and the second equation by μ_i and adding, we obtain

$$(\mu_i + \mu_e)\frac{\partial n}{\partial t} = (D_i \mu_e + D_e \mu_i)\nabla^2 n \tag{5.17}$$

or

$$\boxed{\frac{\partial n}{\partial t} = D_a \nabla^2 n} \tag{5.18}$$

where

$$\boxed{D_a = \frac{\mu_i D_e + \mu_e D_i}{\mu_e + \mu_i}} \quad \text{Ambipolar diffusion coefficient} \tag{5.19}$$

For $T_e = T_i$,

$$D_a \simeq 2D_i \tag{5.20}$$

5.2.2 Diffusion in a Slab

Diffusion Equation:

$$\frac{\partial n}{\partial t} = D_a \nabla^2 n \tag{5.21}$$

Let

$$n(\mathbf{r}, t) = T(t) S(\mathbf{r}) \tag{5.22}$$

$$S \frac{dT}{dt} = D T \nabla^2 S$$

or

$$\frac{1}{T}\frac{dT}{dt} = \frac{D}{S}\nabla^2 S = -\frac{1}{\tau} \tag{5.23}$$

In a slab,

$$\frac{1}{T}\frac{dT}{dt} = \frac{D}{S}\frac{d^2 S}{dx^2} = -\frac{1}{\tau} \tag{5.24}$$

For $T(t)$,

$$\frac{dT}{dt} = -\frac{T}{\tau} \tag{5.25}$$

$$\longrightarrow T = T_0 e^{-\frac{t}{\tau}} \tag{5.26}$$

For $S(x)$,

$$\frac{d^2 S}{dx^2} = -\frac{1}{D\tau} S \tag{5.27}$$

$$\longrightarrow S = A \cos \frac{x}{\sqrt{D\tau}} + B \sin \frac{x}{\sqrt{D\tau}} \tag{5.28}$$

From the boundary condition $n(\pm L, t) = 0$ or $S(\pm L) = 0$,

1. $B = 0$ and
$$\frac{L}{\sqrt{D\tau}} = (l + \frac{1}{2})\pi \quad \text{with} \quad l = 0, 1, 2, \ldots$$
$$\longrightarrow \tau_l = \left[\frac{L}{(l + \frac{1}{2})\pi}\right]^2 \frac{1}{D}$$

2. $A = 0$ and
$$\frac{L}{\sqrt{D\tau}} = m\pi \quad \text{with} \quad m = 1, 2, \ldots$$
$$\longrightarrow \tau_m = \left[\frac{L}{m\pi}\right]^2 \frac{1}{D}$$

Therefore, we obtain

$$S(x) = \begin{cases} A_l \cos \dfrac{(l + \frac{1}{2})\pi x}{L} \\ B_m \sin \dfrac{m\pi x}{L} \end{cases} \tag{5.29}$$

General solution:

$$n(x,t) = \sum_{l=0}^{\infty} a_l e^{-\frac{t}{\tau_l}} \cos \frac{(l + \frac{1}{2})\pi x}{L} + \sum_{m=0}^{\infty} b_m e^{-\frac{t}{\tau_m}} \sin \frac{m\pi x}{L} \tag{5.30}$$

Expansion coefficients can be determined form the initial condition:

$$n(x,0) = \sum_{l=0}^{\infty} a_l \cos \frac{(l+\frac{1}{2})\pi}{L} x + \sum_{m=0}^{\infty} b_m \sin \frac{m\pi}{L} x \tag{5.31}$$

$$\begin{aligned} a_l =& \frac{1}{L}\int_{-L}^{L} n(x,0) \cos \frac{(l+\frac{1}{2})\pi x}{L} dx \\ b_m =& \frac{1}{L}\int_{-L}^{L} n(x,0) \sin \frac{m\pi x}{L} dx \end{aligned} \tag{5.32}$$

NOTES:

- τ increases as L increases or D decreases.
- τ decreases as the mode number (l or m) increases.
  Higher modes decay faster than the lowest (fundamental) mode.
  After sufficiently long time, only the fundamental mode remains.

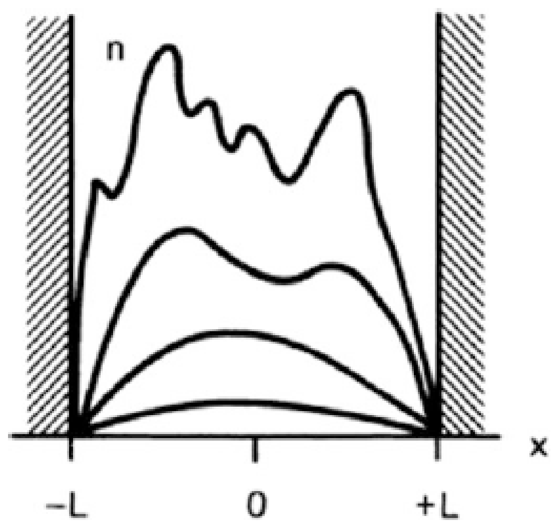

Figure 5.2:

5.2.3 Diffusion in a Cylinder

Diffusion equation:

$$\frac{1}{T}\frac{dT}{dt} = \frac{D}{S}\nabla^2 S = -\frac{1}{\tau}$$

For a infinitely long cylinder ($\frac{\partial}{\partial\theta} = \frac{\partial}{\partial z} = 0$),

$$\nabla^2 S = \frac{d^2 S}{dr^2} + \frac{1}{r}\frac{dS}{dr} = -\frac{1}{D\tau} S$$

Or

$$\frac{d^2 S}{dr^2} + \frac{1}{r}\frac{dS}{dr} + \frac{1}{D\tau} S = 0 \tag{5.33}$$

whose solution is given by

$$S(r) = AJ_0\left(\frac{r}{\sqrt{D\tau}}\right) + BN_0\left(\frac{r}{\sqrt{D\tau}}\right) \tag{5.34}$$

Since $N_0 \longrightarrow \infty$ as $r \longrightarrow 0$, $B = 0$.
From the boundary condition $n(a,t) = 0$ or $S(a) = 0$,

$$S(a) = AJ_0\left(\frac{a}{\sqrt{D\tau}}\right) = 0$$

so that

$$\frac{a}{\sqrt{D\tau}} = \xi_l \quad \text{for} \quad l = 1, 2, 3, \ldots$$

where ξ_l is the lth zero of J_0 ($\xi_1 = 2.405, \xi_2 = 5.520, \xi_3 = 8.654$).

$$\tau_l = \left(\frac{a}{\xi_l}\right)^2 \frac{1}{D} \tag{5.35}$$

$$S(r) = A_l J_0\left(\frac{\xi_l r}{a}\right) \tag{5.36}$$

General Solution:

$$n(r,t) = \sum_{l=1}^{\infty} a_l e^{-\frac{t}{\tau}} J_0\left(\frac{\xi_l r}{a}\right) \tag{5.37}$$

a_l can be found from the initial condition:

$$n(r,0) = \sum_{l=1}^{\infty} a_l J_0\left(\frac{\xi_l r}{a}\right)$$

Use

$$\int_0^1 J_0(\xi r) J_0(\xi' r) r dr = \frac{1}{2}[J_1(\xi)]^2 \delta_{\xi,\xi'},$$

to get

$$a_l = \frac{2}{a^2[J_1(\xi)]^2} \int_0^a n(r,0) J_0\left(\frac{\xi_l r}{a}\right) r dr\,. \tag{5.38}$$

5.3 Steady State Solutions

To maintain a steady state, a source must be added so that the diffusion equation becomes

$$\boxed{\frac{\partial n}{\partial t} - D\nabla^2 n = Q(\mathbf{r})}\,. \tag{5.39}$$

5.3.1 Constant Ionization Function

Ionization is produced by energetic electrons in the tail of the Maxwellian distribution.
The source term is proportional to the electron density: $Q = Zn$.
Z is called the ionization function. Then

$$\nabla^2 n = -\frac{Z}{D} n \tag{5.40}$$

5.3.2 Plane Source

$$\frac{d^2n}{dx^2} = -\frac{Q}{D}\delta(x) \tag{5.41}$$

Except at $x = 0$,

$$\frac{d^2n}{dx^2} = 0$$

Applying boundary conditions:

- $n(\pm L) = 0$
- $n(0) = n_0$

we obtain

$$n(x) = n_0 \left(1 - \frac{|x|}{L}\right) . \tag{5.42}$$

5.3.3 Lince Source

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dn}{dr}\right) = -\frac{Q}{D}\delta(r) \tag{5.43}$$

Applying the boundary condition $n(a) = 0$, we obtain

$$n(r) = n_0 \ln \frac{a}{r} \tag{5.44}$$

5.4 Recombination

The recombination term must be proportional to $n_i n_e = n^2$:
Without the diffusion term, we have

$$\boxed{\frac{\partial n}{\partial t} = -\alpha n^2} \tag{5.45}$$

where α is the recombination coefficient.

$$-\frac{1}{n^2}\frac{\partial n}{\partial t} = \alpha$$

$$d\left(\frac{1}{n}\right) = \alpha dt$$

Or

$$\frac{1}{n} = \alpha t + C$$

$$\frac{1}{n(\mathbf{r}, t)} = \frac{1}{n_0(\mathbf{r})} + \alpha t \tag{5.46}$$

After the density has fallen far below its initial value,

$$\boxed{n \propto \frac{1}{\alpha t}} \tag{5.47}$$

NOTES:

- For high n, recombination dominates: $n \propto \frac{1}{t}$.
- For low n ($|D\nabla^2 n| \gg |\alpha n^2|$), diffusion dominates: $n \propto e^{-t/\tau}$.
-
  - Diffusion gives rise to spatial modes which are approximately sinusoidal in nature.
  - Recombination tends to produce a spatially uniform plasma, since recombination rate depends on the local density and, therefore, recombination acts to flatten any non-uniformities in the plasma density.

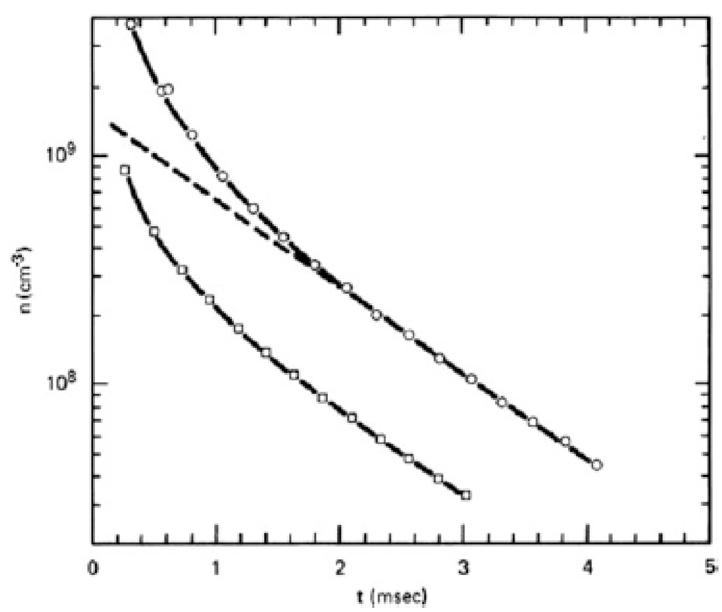

Figure 5.3: