5.5 Diffusion across a Magnetic Field

The fluid equation of motion:

$$mn\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = \pm en(\mathbf{E} + \mathbf{v}\times\mathbf{B}) - \nabla p - mn\nu\mathbf{v} = 0 \tag{5.48}$$

$$\begin{aligned} mn\nu v_x &= \pm enE_x - KT\frac{\partial n}{\partial x} \pm env_yB \\ mn\nu v_y &= \pm enE_y - KT\frac{\partial n}{\partial y} \mp env_xB \\ mn\nu v_z &= \pm enE_z - KT\frac{\partial n}{\partial z} \end{aligned} \tag{5.49}$$

Or

$$\begin{aligned} v_x &= \pm\mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} \pm \frac{\omega_c}{\nu}v_y \\ v_y &= \pm\mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} \mp \frac{\omega_c}{\nu}v_x \\ v_z &= \pm\mu E_z - \frac{D}{n}\frac{\partial n}{\partial z} : \quad \text{same as for } \mathbf{B} = 0 \end{aligned} \tag{5.50}$$

Hence,

$$\begin{aligned} (1+\omega_c^2\tau^2)v_x &= \pm\mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} + \omega_c^2\tau^2\frac{E_y}{B} \mp \omega_c^2\tau^2\frac{KT}{eB}\frac{1}{n}\frac{\partial n}{\partial y} \\ (1+\omega_c^2\tau^2)v_y &= \pm\mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} - \omega_c^2\tau^2\frac{E_x}{B} \pm \omega_c^2\tau^2\frac{KT}{eB}\frac{1}{n}\frac{\partial n}{\partial x} \end{aligned} \tag{5.51}$$

Note that

$$\mathbf{v}_E = \frac{\mathbf{E}\times\mathbf{B}}{B^2} : \qquad \mathbf{E}\times\mathbf{B} \text{ drift}$$

$$\mathbf{v}_D = \frac{KT}{qB}\frac{\mathbf{B}\times\nabla n}{Bn} \quad \text{Diamagnetic drift}$$

or

$$\begin{aligned} v_{Ex} &= \frac{E_y}{B} & v_{Ey} &= -\frac{E_x}{B} \\ v_{Dx} &= \mp\frac{KT}{eB}\frac{1}{n}\frac{\partial n}{\partial y} & v_{Dy} &= \pm\frac{KT}{eB}\frac{1}{n}\frac{\partial n}{\partial x} \end{aligned} \tag{5.52}$$

and define

$$\boxed{\begin{aligned} \mu_\perp &= \frac{\mu}{1+\omega_c^2\tau^2} \\ D_\perp &= \frac{D}{1+\omega_c^2\tau^2} \end{aligned}} \tag{5.53}$$

to obtain

$$\boxed{\begin{aligned} \mathbf{v}_\perp &= \pm\mu_\perp\mathbf{E}_\perp - D_\perp\frac{\nabla n}{n} + \frac{\mathbf{v}_E + \mathbf{v}_D}{1+\nu^2/\omega_c^2} \\ \mathbf{v}_\parallel &= \pm\mu\mathbf{E}_\parallel - D\frac{\nabla n}{n} \end{aligned}} \tag{5.54}$$

NOTE:

- $\mathbf{v}_E$ and $\mathbf{v}_D$: perpendicular to the gradient in potential and density.
  The mobility and diffusion drifts: parallel to the gradient in potential and density.
  But these drifts are slowed down by the factor of $1 + \omega_c^2\tau^2$.

- When $\omega_c^2\tau^2 \ll 1$, the magnetic field has little effect on diffusion.
  When $\omega_c^2\tau^2 \gg 1$, the magnetic field significantly retard the diffusion rate across $\mathbf{B}$.

- When $\omega_c^2\tau^2 \gg 1$,

$$D_\perp = \frac{KT}{m\nu}\frac{1}{\omega_{ce}^2\tau^2} = \frac{KT\nu}{m\omega_c^2}.$$

  Comparing with

$$D_\parallel = \frac{KT}{m\nu}$$

  we note

  - $D_\parallel \propto \nu^{-1}$: Collisions retard the motion.
    $D_\perp \propto \nu$: Collisions are needed for cross-field migration.
  - $D_\parallel \propto m^{-\frac{1}{2}}$: Electrons move faster. ($\nu \sim m^{-1/2}$)
    $D_\perp \propto m^{\frac{1}{2}}$: Electrons excape more slowly because of their small Larmor radius.
  - $D_\parallel = \frac{KT}{m\nu} \sim v_{th}^2\tau \sim \frac{\lambda_m^2}{\tau}$
    Diffusion is a random-walk process with a step length λ_m.
    $D_\perp = \frac{KT\nu}{m\omega_c^2} \sim v_{th}^2\frac{r_L^2}{v_{th}^2}\nu \sim \frac{r_L^2}{\tau}$
    Diffusion is a random-walk process with a step length r_L.

5.6 Collisions in Fully Ionized Plasmas

5.6.1 Plasma Resisitivity

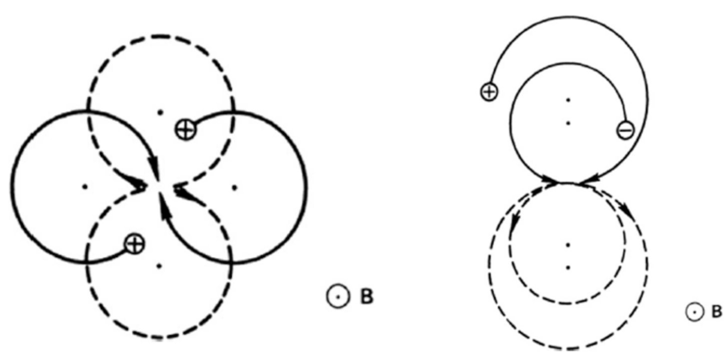

Figure 5.4: (left) Shift of guiding centers of two like particles making a 90° collision. (right) Shift of guiding centers of two oppositely charged particles making a 180° collision.

The fluid equations of motion are

$$m_i n\frac{d\mathbf{v}_i}{dt} = en(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) - \nabla p_i - \nabla\cdot\boldsymbol{\pi}_i + \mathbf{P}_{ie} \tag{5.55}$$

$$m_e n \frac{d\mathbf{v}_e}{dt} = -en(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) - \nabla p_e - \nabla \cdot \boldsymbol{\pi}_e + \mathbf{P}_{ei} \tag{5.56}$$

where $\mathbf{P}_{ie}$ is the change in ion momentum due to collisions with electrons. From the conservation of momentum,

$$\mathbf{P}_{ei} = -\mathbf{P}_{ie} \tag{5.57}$$

$$\mathbf{P}_{ei} = m_e n(\mathbf{v}_i - \mathbf{v}_e)\nu_{ei} \tag{5.58}$$

For Coulomb collisions,

$$\mathbf{P}_{ei} \propto e^2, n_e, n_i, \mathbf{v}_i - \mathbf{v}_e$$

or

$$\mathbf{P}_{ei} = \eta e^2 n^2 (\mathbf{v}_i - \mathbf{v}_e)\,. \tag{5.59}$$

Therefore, we obtain

$$\nu_{ei} = \frac{ne^2}{m_e}\eta \tag{5.60}$$

Let $B = 0$ and $KT_e = 0$ so that $\nabla \cdot \mathsf{P} = 0$. Then in steady state,

$$en\mathbf{E} = \mathbf{P}_{ei} \tag{5.61}$$

Since $\mathbf{J} = ne(\mathbf{v}_i - \mathbf{v}_e)$,

$$\mathbf{P}_{ei} = \eta ne\mathbf{J} \tag{5.62}$$

It follows that

$$\mathbf{E} = \eta\mathbf{J}: \quad \text{Ohm's law.} \tag{5.63}$$

5.6.2 Coulomb Collisions

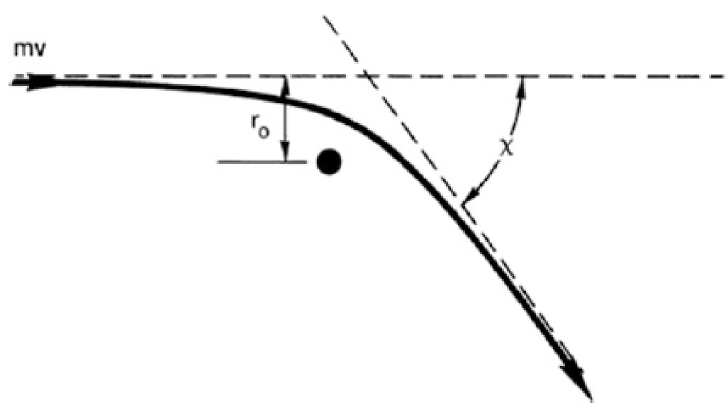

Figure 5.5: Here, r_0 is called the impact parameter.

The Coulomb force given by

$$F = -\frac{e^2}{4\pi\epsilon_0 r^2} \tag{5.64}$$

is felt during the time the electron is in the vicinity of the ion: this time is roughly

$$T \simeq \frac{r_0}{v} \tag{5.65}$$

The change in the electron's momentum is

$$\triangle(mv) = |FT| \simeq \frac{e^2}{4\pi\epsilon_0 r_0 v} \tag{5.66}$$

For a 90° collision,

$$\triangle(mv) \simeq mv \simeq \frac{e^2}{4\pi\epsilon_0 r_0 v}$$

so

$$r_0 \simeq \frac{e^2}{4\pi\epsilon_0 mv^2} \tag{5.67}$$

The cross section is then

$$\sigma = \pi r_0^2 \simeq \frac{e^4}{16\pi\epsilon_0^2 m^2 v^4} \tag{5.68}$$

The collision frequency is

$$\nu_{ei} = n\sigma v \simeq \frac{ne^4}{16\pi\epsilon_0^2 m^2 v^3} \tag{5.69}$$

and the resistivity is

$$\eta = \frac{m\nu_{ei}}{ne^2} = \frac{e^2}{16\pi\epsilon_0^2 mv^3} \tag{5.70}$$

Replacing v^2 with KT_e/m for a Maxwellian plasma, we obtain

$$\eta = \frac{\pi e^2 m^{\frac{1}{2}}}{(4\pi\epsilon_0)^2 (KT_e)^{3/2}} \tag{5.71}$$

This resistivity is based on large-angle collisions alone.
In practice,

$$\eta = \frac{\pi e^2 m^{\frac{1}{2}}}{(4\pi\epsilon_0)^2 (KT_e)^{3/2}} \ln\Lambda \tag{5.72}$$

where

$$\Lambda = \left\langle \frac{\lambda_D}{r_0} \right\rangle \tag{5.73}$$

which represents the maximum impact parameter averaged over a Maxwellian distribution.
NOTES:

- η is independent of n(except for the weak dependence in $\ln\Lambda$).
  But in a weakly ionized plasma, η depends on n.
  ($\mathbf{J} = -ne\mathbf{v}_e$, $\mathbf{v}_e = -\mu_e\mathbf{E}$ so that $\mathbf{J} = ne\mu_e\mathbf{E}$.)
- $\eta \propto (KT_e)^{-3/2}$: Good conductor at high temperature.
  Ohmic heating ($J^2\eta$) becomes ineffective as temperature increases.
- $\nu_{ei} \propto v^{-3}$:
  - The current is mainly carried by the fast electrons.
  - Electron runaway can occur when an electric field is suddely applied.

- Numerical values of η:

copper	$\eta = 2\times 10^{-8}$ohm-m
stainless steel	$\eta = 7\times 10^{-7}$ohm-m
mecury	$\eta = 1\times 10^{-6}$ohm-m
100 eV hydrogen plasma	$\eta = 5\times 10^{-7}$ohm-m

5.7 Magnetohydrodynamics

Define

$$\begin{aligned} \rho_m &\equiv n_i m_i + n_e m_e \simeq n(m_i+m_e) \\ \mathbf{V} &\equiv \frac{n_i m_i \mathbf{v}_i + n_e m_e \mathbf{v}_e}{n_i m_i + n_e m_e} \simeq \frac{m_i\mathbf{v}_i + m_e\mathbf{v}_e}{m_i+m_e} \\ \mathbf{J} &\equiv e(n_i\mathbf{v}_i - n_e\mathbf{v}_e) \simeq ne(\mathbf{v}_i-\mathbf{v}_e) \end{aligned} \tag{5.74}$$

5.7.1 Continuity Equation

From the continuity equations

$$\frac{\partial n_i}{\partial t} + \nabla\cdot(n_i\mathbf{v}_i) = 0 \tag{5.75}$$

$$\frac{\partial n_e}{\partial t} + \nabla\cdot(n_e\mathbf{v}_e) = 0 \tag{5.76}$$

we obtain the continuity equation for mass ρ_m

$$\frac{\partial}{\partial t}(n_i m_i + n_e m_e) + \nabla\cdot(n_i m_i\mathbf{v}_i + n_e m_e\mathbf{v}_e) = 0 \tag{5.77}$$

or

$$\boxed{\frac{\partial \rho_m}{\partial t} + \nabla\cdot(\rho_m\mathbf{V}) = 0} \tag{5.78}$$

5.7.2 Momentum Equation

Fluid equations of motion are (neglecting quadratic terms in $\mathbf{v}$)

$$n_i m_i\frac{\partial \mathbf{v}_i}{\partial t} = en_i(\mathbf{E}+\mathbf{v}_i\times\mathbf{B}) - \nabla p_i + \mathbf{P}_{ie} \tag{5.79}$$

$$n_e m_e\frac{\partial \mathbf{v}_e}{\partial t} = -en_e(\mathbf{E}+\mathbf{v}_e\times\mathbf{B}) - \nabla p_e + \mathbf{P}_{ei} \tag{5.80}$$

Eq. (5.79) + Eq. (5.80):

$$\frac{\partial}{\partial t}(n_i m_i\mathbf{v}_i + n_e m_e\mathbf{v}_e) = e(n_i\mathbf{v}_i - n_e\mathbf{v}_e)\times\mathbf{B} - \nabla p \tag{5.81}$$

$$\boxed{\rho_m\frac{\partial \mathbf{V}}{\partial t} = \mathbf{J}\times\mathbf{B} - \nabla p} \tag{5.82}$$

where $p = p_e + p_i$.

5.7.3 Ohm's Law

$m_e \times$ Eq. (5.79) $- m_i \times$ Eq. (5.80):

$$nm_i m_e \frac{\partial}{\partial t}(\mathbf{v}_i - \mathbf{v}_e) = en(m_i + m_e)\mathbf{E} + en(m_e\mathbf{v}_i + m_i\mathbf{v}_e)\times\mathbf{B} - m_e\nabla p_i - m_i\nabla p_e - (m_i + m_e)\mathbf{P}_{ei} \tag{5.83}$$

$$\frac{m_i m_e}{e}\frac{\partial}{\partial t}\mathbf{J} = e\rho_m\mathbf{E} - (m_i + m_e)ne\eta\mathbf{J} - m_e\nabla p_i + m_i\nabla p_e + en(m_e\mathbf{v}_i + m_i\mathbf{v}_e)\times\mathbf{B} \tag{5.84}$$

where we have used that $n_i \simeq n_e = n$.
Since

$$m_e\mathbf{v}_i + m_i\mathbf{v}_e = m_i\mathbf{v}_i + m_e\mathbf{v}_e + m_i(\mathbf{v}_e - \mathbf{v}_i) + m_e(\mathbf{v}_i - \mathbf{v}_e)$$

$$= \frac{\rho_m}{n}\mathbf{V} - (m_i - m_e)\frac{\mathbf{J}}{ne},$$

Eq. (5.84)$\times\frac{1}{e\rho_m}$ becomes

$$\mathbf{E} + \mathbf{V}\times\mathbf{B} - \eta\mathbf{J} = \frac{1}{e\rho_m}\left[\frac{m_i m_e}{e}\frac{\partial\mathbf{J}}{\partial t} + (m_i - m_e)\mathbf{J}\times\mathbf{B} + m_e\nabla p_i - m_i\nabla p_e\right] \tag{5.85}$$

In the limit $m_e/m_i \longrightarrow 0$,

$$\mathbf{E} + \mathbf{V}\times\mathbf{B} = \frac{m_e}{ne^2}\frac{\partial\mathbf{J}}{\partial t} + \eta\mathbf{J} + \frac{1}{en}(\mathbf{J}\times\mathbf{B} - \nabla p_e) \tag{5.86}$$

If

$$\left|\frac{\frac{m_e}{ne^2}\frac{\partial\mathbf{J}}{\partial t}}{\frac{\mathbf{J}\times\mathbf{B}}{en}}\right| = \frac{\frac{m_e\omega}{e}}{B} = \frac{\omega}{\omega_{ce}} \ll 1 \tag{5.87}$$

we obtain

$$\boxed{\mathbf{E} + \mathbf{V}\times\mathbf{B} = \eta\mathbf{J} + \frac{1}{en}(\mathbf{J}\times\mathbf{B} - \nabla p_e)} \tag{5.88}$$

This is called the *generalized Ohm's law.*

If

$$\left|\frac{\frac{1}{ne}\mathbf{J}\times\mathbf{B}}{\eta\mathbf{J}}\right| = \frac{\frac{B}{ne}}{\eta} = \frac{\frac{B}{ne}}{\frac{m_e}{ne^2}\nu_{ei}} = \frac{\omega_{ce}}{\nu_{ei}} \ll 1 \tag{5.89}$$

and

$$\left|\frac{\nabla p_e}{n}\right| \simeq |\nabla KT_e| \ll |e\mathbf{E}| \tag{5.90}$$

we have

$$\boxed{\mathbf{E} + \mathbf{V}\times\mathbf{B} = \eta\mathbf{J}} \tag{5.91}$$

or

$$\boxed{\mathbf{J} = \sigma\left(\mathbf{E} + \mathbf{V}\times\mathbf{B}\right)} \tag{5.92}$$

NOTE:

- Ohm's law, which relates the current density **J** and the electric field **E**, is

$$\mathbf{J} = \sigma \mathbf{E} \tag{5.93}$$

Here **E** is the total electric field and must include the electric field induced by the motion of the fluid across the magnetic field. Ohm's law then becomes

$$\mathbf{J} = \sigma \left(\mathbf{E} + \mathbf{V} \times \mathbf{B} \right) \tag{5.94}$$

where **V** is the fluid velocity. It is an approximation of a generalized Ohm's law.

- When collisions vanish, the conductivity becomes infinite. In order to have finite current, we must have for an ideal MHD fluid

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0 \tag{5.95}$$

or

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} \tag{5.96}$$

- The displacement current can be neglected in MHD theory.

$$\left| \frac{\partial \mathbf{D}}{\partial t} \right| = \epsilon_0 \left| \frac{\partial \mathbf{E}}{\partial t} \right| \sim \epsilon_0 \frac{E}{T} \tag{5.97}$$

On the other hand,

$$J = \left| \frac{1}{\mu_0} \nabla \times \mathbf{B} \right| \sim \frac{B}{\mu_0 L} \tag{5.98}$$

Using $\mathbf{E} \simeq \mathbf{V} \times \mathbf{B}$, the ratio of two terms is

$$\frac{\left| \frac{\partial \mathbf{D}}{\partial t} \right|}{|\mathbf{J}|} \sim \frac{\frac{\epsilon_0 E}{T}}{\frac{B}{\mu_0 L}} \sim \epsilon_0 \mu_0 V \frac{L}{T} \sim \frac{V^2}{c^2} \ll 1 \tag{5.99}$$

5.7.4 Equation of State

Assuming the fluid is adiabatic

$$\boxed{\frac{d}{dt} \left(p \rho_m^{-\gamma} \right) = 0} \tag{5.100}$$

where

$$\gamma = \frac{C_p}{C_V}$$

If the fluid is isothermal, let $\gamma = 1$.

$$y = \frac{r_{Li}}{a}$$

$$x = \left(\frac{m_i}{m_e}\right)^{1/2} \frac{V_{T_i}\tau_{ii}}{a}$$

$$\frac{|\eta \mathbf{J}|}{|\mathbf{v} \times \mathbf{B}|} \sim \frac{(m_e/m_i)^{1/2}}{\omega \tau_{ii}} \left(\frac{r_{Li}}{a}\right)^2 \ll 1$$

(1) High collisionality $x \ll 1$
(2) Small gyro radius $y \ll 1$
(3) Small resistivity $y^2/x \ll 1$

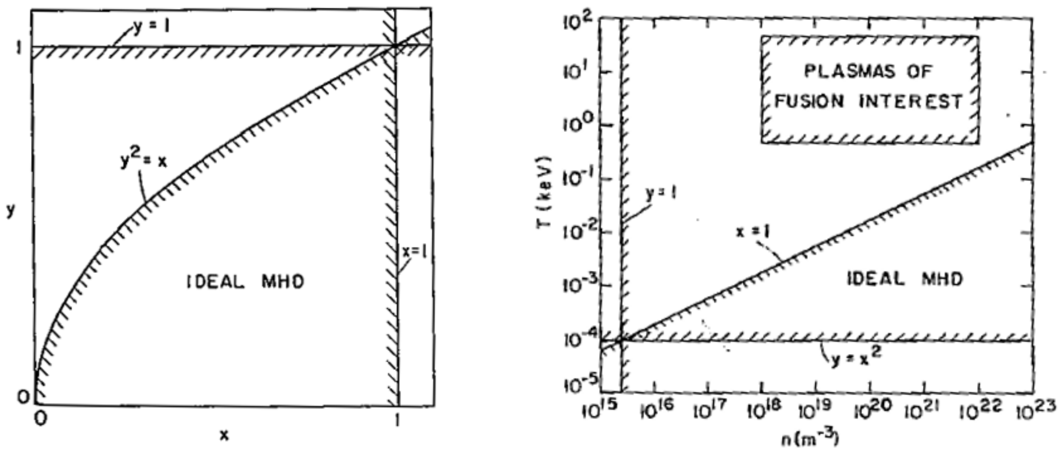

Figure 5.6: Region of validity of the ideal MHD model.

5.7.5 The MHD equations

Continuity equation:

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{V}) = 0 \tag{5.101}$$

Equation of motion:

$$\rho_m \frac{\partial \mathbf{V}}{\partial t} = \mathbf{J} \times \mathbf{B} - \nabla p \tag{5.102}$$

Ohm's law:

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \tag{5.103}$$

An equation of state:

$$\frac{d}{dt}(p\rho_m^{-\gamma}) = 0 \tag{5.104}$$

Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5.105}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{5.106}$$

5.7.6 Energy Equation (option)

From the momentum equation and Mawell's equation

$$\rho_m \frac{d\mathbf{V}}{dt} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p \tag{5.107}$$

Taking the dot product of the this equation with $\mathbf{V}$,

$$\rho_m \mathbf{V} \cdot \frac{d\mathbf{V}}{dt} = \frac{1}{\mu_0}\mathbf{V} \cdot (\nabla \times \mathbf{B}) \times \mathbf{B} - \mathbf{V} \cdot \nabla p \tag{5.108}$$

The term on the left hand side can be written as

$$\begin{aligned} \rho_m \mathbf{V} \cdot \frac{d\mathbf{V}}{dt} =& \rho_m \mathbf{V} \cdot \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla\right) \mathbf{V} \\ =& \rho_m \frac{\partial}{\partial t}\left(\frac{V^2}{2}\right) + \rho_m \mathbf{V} \cdot \nabla \left(\frac{V^2}{2}\right) \\ =& \frac{\partial}{\partial t}\left(\frac{1}{2}\rho_m V^2\right) - \frac{1}{2}V^2 \frac{\partial \rho_m}{\partial t} + \frac{1}{2}\rho_m \mathbf{V} \cdot \nabla V^2 \end{aligned} \tag{5.109}$$

Use the continuity equation

$$\frac{\partial \rho_m}{\partial t} = -\nabla \cdot (\rho_m \mathbf{V})$$

to obtain

$$\rho_m \mathbf{V} \cdot \frac{d\mathbf{V}}{dt} = \frac{\partial}{\partial t}\left(\frac{1}{2}\rho_m V^2\right) + \nabla \cdot \left[\left(\frac{1}{2}\rho_m V^2\right)\mathbf{V}\right]. \tag{5.110}$$

For the second term on the right hand side, use the equation of state

$$\frac{d}{dt}(p\rho_m^{-\gamma}) = 0 \tag{5.111}$$

or

$$\rho_m^{-\gamma}\frac{dp}{dt} - \gamma p \rho_m^{-(\gamma+1)}\frac{d\rho_m}{dt} = 0 \tag{5.112}$$

which reduces to

$$\frac{dp}{dt} - \frac{\gamma p}{\rho_m}\frac{d\rho_m}{dt} = 0\,. \tag{5.113}$$

Since

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + (\mathbf{V} \cdot \nabla)p$$

$$\frac{d\rho_m}{dt} = -\rho_m(\nabla \cdot \mathbf{V}),$$

$$\frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{V} = 0 \tag{5.114}$$

or

$$\frac{\partial p}{\partial t} + (1-\gamma)\mathbf{V} \cdot \nabla p + \gamma \nabla \cdot (p\mathbf{V}) = 0 \tag{5.115}$$

The first term on the right hand side can be rewritten as

$$\begin{aligned}\frac{1}{\mu_0}\mathbf{V}\cdot(\nabla\times\mathbf{B})\times\mathbf{B} &= -\frac{1}{\mu_0}(\mathbf{V}\times\mathbf{B})\cdot(\nabla\times\mathbf{B}) \\ &= \frac{1}{\mu_0}\mathbf{E}\cdot(\nabla\times\mathbf{B})\end{aligned} \tag{5.116}$$

where we have assumed $\mathbf{E} = -\mathbf{V}\times\mathbf{B}$, i.e., the plasma is perfectly conducting ($\sigma \longrightarrow \infty$). Now use the relation

$$\nabla\cdot(\mathbf{E}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{B}) \tag{5.117}$$

to get

$$\frac{1}{\mu_0}\mathbf{V}\cdot(\nabla\times\mathbf{B})\times\mathbf{B} = -\frac{1}{2\mu_0}\frac{\partial B^2}{\partial t} - \frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B})\,. \tag{5.118}$$

Combining all terms, we obtain the energy conservation relation for an adiabatic MHD fluids as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_m V^2 + \frac{p}{\gamma-1} + \frac{B^2}{2\mu_0}\right) + \nabla\cdot\left(\frac{1}{2}\rho_m V^2\mathbf{V} + \frac{\gamma}{\gamma-1}p\mathbf{V} + \frac{\mathbf{E}\times\mathbf{B}}{\mu_0}\right) = 0 \tag{5.119}$$

Integrating over the entire fluid-plus-vacuum volume, the divergence term yields a surface integral which vanishes. Hence we obatin the energy conservation law

$$\boxed{\int\left(\frac{1}{2}\rho_m V^2 + \frac{p}{\gamma-1} + \frac{B^2}{2\mu_0}\right)dv = \text{const}} \tag{5.120}$$

Or

$$K + W = \text{const} \tag{5.121}$$

where

$$K = \int \frac{1}{2}\rho_m V^2 dv \quad \text{kinetic energy} \tag{5.122}$$

$$W = \int\left(\frac{p}{\gamma-1} + \frac{B^2}{2\mu_0}\right)dv \quad \text{potential energy} \tag{5.123}$$