Eigen-emittance and Beam matching

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A few words on beam matching

A beam matched to a periodic focusing system has envelope oscillations of minimum amplitude. Furthermore, we shall find in Section 4.4 that the emittance growth caused by lens non-linearities is smallest for a matched beam. We outlined a numerical method to find matched beam distributions in Section 3.7. In this section, we shall study analytic methods that use

In a particle simulation involving a periodic lattice, it is usually desired to generate particles in a matched state, which means that the shape of the distribution should not change after one passage through the lattice. In fact, if a matched distribution can be found, one often has already accomplished a great deal in the understanding of the simulation. Additionally, there are circumstances in which the knowledge of the effective emittance and optics parameters

The matched beam envelope is the solution to the KV envelope equations with the periodicity of the focusing lattice. The matched solution is generally believed to have the smallest maximum radial excursions relative to other possible envelope evolutions in the lattice and it requires particular initial conditions in the envelope of beam

[Stanley Humphries, Jr.]  
[Malte Titze]  
[Steve Lund]
Continuing the discussion of periodic beam lines, the next step is to introduce the concept of a matched distribution. A matched distribution at any point in a periodic beam line is a phase space distribution of particles that is unchanged after the bunch is transported along one periodic section of the beam line. For the present purposes, we need consider only the second-order moments of the distribution: we do not need to specify whether the distribution is uniform, parabolic, Gaussian or some more exotic function. For convenience, we define the $2n \times 2n$ matrix $\Sigma$ (in $n$ degrees of freedom) with elements $\Sigma_{ij}$ defined by:

$$\Sigma_{ij} = \langle x_i x_j \rangle, \quad (5.105)$$

$$R \Sigma R^T = \sum_k N^{-1} T^k (N^{-1})^T \epsilon_k = \Sigma. \quad (5.122)$$

Therefore, a matrix $\Sigma$ constructed using (5.116) is unchanged under transport through one periodic section of the beam line. In other words, such a matrix represents a matched distribution. Note that this is true for any values of the emittances $\epsilon_k$: there are infinitely many matched distributions for a given beam line, although the number of degrees of freedom in choosing a matched distribution is only equal to the number of degrees of freedom in the particle motion.
The so-called Courant-Snyder invariant:

\[ I_{CS} = \gamma(s)x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s) \]

\[ = \begin{pmatrix} x & x' \end{pmatrix} \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} \]

\[ = \begin{pmatrix} x \\ x' \end{pmatrix}^T \begin{pmatrix} 1/w & 0 \\ -w' & w \end{pmatrix}^T \begin{pmatrix} 1/w & 0 \\ -w' & w \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} \]

\[ = \bar{z}^T \bar{z} = \text{const.} \]

\[ \bar{z} = \begin{pmatrix} \bar{x} \\ \bar{x}' \end{pmatrix} = \text{normalized coordinates} \]

\[ = \begin{pmatrix} 1/w & 0 \\ -w' & w \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} \]

\[ = Q(s)z(s) \]

\[ \neq \text{const.} \]
Normalized Coordinates

\[ M(s) = \begin{bmatrix} \sqrt{\frac{\beta}{\beta_0}}(\cos \psi + \alpha_0 \sin \psi) & \sqrt{\beta_0 \beta} \sin \psi \\ \frac{1 + \alpha_0 \alpha}{\sqrt{\beta_0 \beta}} \sin \psi + \frac{\alpha_0 - \alpha}{\sqrt{\beta_0 \beta}} \cos \psi & \sqrt{\beta_0 \beta} (\cos \psi - \alpha \sin \psi) \end{bmatrix} \]

\[ = \begin{bmatrix} w & 0 & w^{-1} \\ w' & w^{-1} & 0 \end{bmatrix} \begin{bmatrix} \cos \psi & \sin \psi & w_0^{-1} \\ -\sin \psi & \cos \psi & 0 \end{bmatrix} \begin{bmatrix} x' \\ x \\ s \end{bmatrix} \]

\[ \text{Radius} = A = \sqrt{I_{CS}} \]
Another Invariant

By counter-acting the rotation, we can make the coordinates unchanged:

\[
\begin{pmatrix}
\bar{x} \\
\bar{x'}
\end{pmatrix} = \bar{z} = P(s)\bar{z}(s) = P(s)Q(s)z(s) = \text{const.} = P_0Q_0\bar{z}_0 = P_0\bar{z}_0 = \bar{z}_0 = \begin{pmatrix}
\bar{x}_0 \\
\bar{x'}_0
\end{pmatrix}
\]

Here,

\[
P(s) = \begin{pmatrix}
\cos \psi(s) & -\sin \psi(s) \\
\sin \psi(s) & \cos \psi(s)
\end{pmatrix}, \quad \psi' = \frac{1}{w^2}
\]

Phase advance matrix \hspace{1cm} \text{Phase advance rate}

(\text{CCW rotation})

\[
P(0) = P_0 = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

No need to counter-act the rotation at \(s = 0\)
Proof of the statement in the previous page:

\[ x(s) = Aw(s) \cos[\psi(s) + \phi_0] \]

\[ \rightarrow \] By directly insert this in the coordinate transformation,

\[
\begin{pmatrix}
    \bar{x} \\
    \bar{x}'
\end{pmatrix}
= \begin{pmatrix}
    \cos \psi \times w^{-1} x - \sin \psi \times (-w'x + wx) \\
    \sin \psi \times w^{-1} x + \cos \psi \times (-w'x + wx)
\end{pmatrix}

= \begin{pmatrix}
    A \cos \phi_0 \\
    -A \sin \phi_0
\end{pmatrix}
= \text{const.}

Hence,

\[ \bar{z} = P(s)Q(s)z(s) = \text{const}. \]
New Form of Invariant

Therefore,

\[ I_{CS} = \bar{z}^T \bar{z} = \text{const.} \]

In fact, there can exist other "Quadratic" Invariants, such as

\[ I_\xi = \bar{z}^T \xi \bar{z} = \text{const.} \geq 0 \]

\[ \xi = \text{a } 2 \times 2 \text{ constant positive definite (symmetric) matrix} \]
1) Any positive-definite (because it should represent particle counts) distribution function formed from a set of single-particle constants of the motion \((C_i)\) will produce a valid, exact equilibrium solution to the Vlasov equation [Seteev lund]:

\[
\frac{d}{ds} f\{\{C_i\}\} = 0
\]

2) However, the Gaussian distribution is commonly used [S. Y. Lee]:

\[
f(z) = \frac{1}{(2\pi)^{D/2} |\Sigma|} \times \exp \left\{-\frac{1}{2} z^T \Sigma^{-1} z \right\}, \quad \Sigma = \langle zz^T \rangle = \text{beam (covariant) matrix}
\]

D: dimension of \(z\)

From 1) & 2)

\[
\rightarrow \frac{1}{(2\pi)^{D/2} |\Sigma|} \times \exp \left\{-\frac{1}{2} I_\xi \right\}, \quad I_\xi = \bar{z}^T \xi \bar{z} = z^T Q^T P^T \xi P Q z
\]

\[
\rightarrow \Sigma^{-1} = Q^T P^T \xi P Q, \quad \text{or} \quad \Sigma = Q^{-1} P^{-1} \varepsilon P^{-T} Q^{-T}, \quad \text{with} \quad \varepsilon = \xi^{-1}
\]
Beam Distribution in Normalized Coordinates

At $s = 0$

$$\tilde{\Sigma}(0) = \langle \tilde{z}\tilde{z}^T \rangle_0$$

$$= Q_0 \langle z\overline{z}^T \rangle_0 Q_0^T$$

$$= Q_0 \Sigma(0) Q_0^T$$

$$\rightarrow Q_0 \left[ Q_0^{-1} P_0^{-1} \varepsilon P_0^{-T} Q_0^{-T} \right] Q_0^T$$

$$= \varepsilon$$

Without loss of generality, (phase advance is measured from $s=0$)

At $s > 0$

$$\tilde{\Sigma}(s) = \langle \tilde{z}\tilde{z}^T \rangle$$

$$= P^{-1} \tilde{\Sigma}(0) P^{-T}$$

$$= P^{-1} \varepsilon P^{-T}$$

$$\varepsilon = \varepsilon \begin{pmatrix} \bar{\beta} & 0 \\ 0 & 1/\bar{\beta} \end{pmatrix}$$

Without loss of generality, $\varepsilon$ can be written in terms of diagonal matrix:

$\rightarrow$ we choose $s=0$ when initial ellipse is upright, and then apply rotation

$\rightarrow$ two parameters are required to define upright ellipse

$P_0 = I$
Without any filamentation:

\[
\det [\bar{\Sigma}(s)] = \det [P^{-1}\varepsilon P^{-T}]
= \det [\varepsilon]
= \varepsilon^2
\]

With filamentation:

1) If \(\bar{\beta} = 1\), beam distribution is not affected by the phase advance

\[
\bar{\Sigma}(s) = \varepsilon I \quad \rightarrow \quad \det [\bar{\Sigma}(s)] = \varepsilon^2
\]

2) If \(\bar{\beta} \neq 1\)

\[
\bar{\Sigma}(s) = \varepsilon \begin{bmatrix}
\bar{\beta} \cos^2 \psi + \frac{1}{\bar{\beta}} \sin^2 \psi & -\frac{\cos \psi \sin \psi}{\bar{\beta}} + \bar{\beta} \cos \psi \sin \psi \\
-\frac{\cos \psi \sin \psi}{\bar{\beta}} + \bar{\beta} \cos \psi \sin \psi & \bar{\beta} \sin^2 \psi + \frac{1}{\bar{\beta}} \cos^2 \psi
\end{bmatrix}
\]
After Filamentation

\[ \Sigma(s) = \epsilon \left[ \begin{array}{cc} \bar{\beta} \cos^2 \psi + \frac{1}{\beta} \sin^2 \psi & -\frac{\cos \psi \sin \psi}{\beta} + \bar{\beta} \cos \psi \sin \psi \\ -\frac{\cos \psi \sin \psi}{\beta} + \bar{\beta} \cos \psi \sin \psi & \bar{\beta} \sin^2 \psi + \frac{1}{\beta} \cos^2 \psi \end{array} \right] \]

\[ \rightarrow \epsilon \left[ \begin{array}{cc} \bar{\beta} \frac{1}{2} + \frac{1}{\beta} \frac{1}{2} & 0 \\ 0 & \bar{\beta} \frac{1}{2} + \frac{1}{\beta} \frac{1}{2} \end{array} \right] \]

Average over randomly-distributed \( \psi \)

\[ \sqrt{\text{det} [\Sigma(s)]} = \epsilon \frac{1}{2} \left( \bar{\beta} + \frac{1}{\bar{\beta}} \right) \geq \epsilon \sqrt{\bar{\beta} \frac{1}{\bar{\beta}}} = \epsilon \]

Equality for \( \bar{\beta} = 1 \)

\[ \Sigma = Q^{-1} P^{-1} \epsilon P^{-T} Q^{-T} \rightarrow Q^{-1} \epsilon I Q^{-T} \]
Example: $\bar{\beta} \neq 1$
[Example: $\bar{\beta} = 1$]

Normalized coordinates

Laboratory coordinates
Periodic Matching VS Mismatching

→ Plots in the previous page were made with periodically mis-matched launching condition

Periodically matched solution has minimum radial excursion
[Example: $\bar{\beta} = 1$ and periodic matching]
In the periodic focusing system, the particle distribution is non-stationary, however, when plotted in trace space once per period (i.e., in the Poincaré plot), we can treat the beam in stationary equilibrium.

\[ f(s) = f(s + L) \quad \rightarrow \quad \Sigma(s) = \Sigma(s + L) \]
\[ \rightarrow \quad Q(s) = Q(s + L) \]
\[ \rightarrow \quad w(s) = w(s + L) \]
Beam parameterization and invariants in a periodic solenoidal channel*

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• Beam matching in 2+ D
For 2+ Dimension Case

\[ Q^{-1} \begin{pmatrix} \bar{x} \\ \bar{x}' \end{pmatrix} \rightarrow \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} \]

(Norm. frame 1)

Due to coupling

\[ Q^{-1} \begin{pmatrix} \bar{y} \\ \bar{y}' \end{pmatrix} \rightarrow \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} \]

(Norm. frame 2)

CW Rotation

Radius = \( A_1 \)

Radius = \( A_2 \)
Williamson’s Theorem

Diagonalization of an every $2n \times 2n$ real, symmetric, positive definite matrix

$$\Sigma = SDS^T = S\begin{bmatrix} \epsilon_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon_2 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \end{bmatrix}S^T$$

$S^TJS = J, SJS^T = J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

A symplectic matrix unique up to a unitary matrix (a symplectic rotation)

$\Rightarrow$ But, not every unitary matrix can be used here
$\Rightarrow$ The unitary matrix should have a special form. See slide 29

$U(n) = Sp(2n, R) \cap O(2n)$

$S\begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix}S^T = S\begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix}S^T = SU\begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix}U^T S^T$

$\Lambda = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \ddots \end{bmatrix}$

Diagonal elements are “symplectic eigenvalues” (symplectic spectrum) of $\Sigma$

$\det [J\Sigma \pm i\lambda I] = 0$
Eigen-emittance

\[ \det[\Sigma] = \det[S DS^T] = \det[S] \det[D] \det[S^T] = \det[D] = (\epsilon_1 \epsilon_2)^2 \]

\[
\begin{align*}
\text{tr}[(\Sigma J)^2] & = \text{tr}[S DS^T J \cdot S DS^T J] \\
& = \text{tr}[SD \cdot S^T JS \cdot DS^T J] \\
& = \text{tr}[SD \cdot J \cdot DS^T J] \\
& = \text{tr}[DJ \cdot DS^T JS] \\
& = \text{tr}[DJ \cdot DJ] \\
& = \text{tr}[(DJ)^2] \\
& = -2(\epsilon_1^2 + \epsilon_2^2)
\end{align*}
\]

\[\epsilon_{1,2} = \frac{1}{2} \sqrt{-\text{tr}[(DJ)^2] \pm \sqrt{\text{tr}^2[(DJ)^2] - 16 \det[D]}}\]

Invariant under symplectic transformation
From Fischer’s inequality:

\[
\Sigma = \begin{bmatrix} A & C \\ C^T & B \end{bmatrix} \quad \rightarrow \quad \det[\Sigma] \leq \det[A] \det[B]
\]

\[
\rightarrow \quad (\epsilon_1 \epsilon_2)^2 \leq \epsilon_{\text{rms},x}^2 \times \epsilon_{\text{rms},y}^2
\]

From direct calculation (e.g., with the help of Mathematica):

\[
-\frac{1}{2} \text{tr}[(\Sigma J)^2] = \det[A] + \det[B] + 2 \det[C]
\]

\[
= \epsilon_{\text{rms},x}^2 + \epsilon_{\text{rms},y}^2 + 2 \begin{vmatrix} \langle xy \rangle & \langle xy' \rangle \\ \langle x' y \rangle & \langle x' y' \rangle \end{vmatrix}
\]

\[
= \epsilon_1^2 + \epsilon_2^2
\]

* In this slide, we use the notation of \( z = (x, x', y, y')^T \)
For a round beam in solenoids with finite average canonical angular momentum:
(i.e., distribution function is independent of angle is X-Y plane)

Using canonical coordinates

\[
\Sigma = \begin{bmatrix}
\langle X^2 \rangle & \langle XX' \rangle & 0 & \langle XY' \rangle \\
\langle XX' \rangle & \langle X'^2 \rangle & -\langle XY' \rangle & 0 \\
0 & -\langle XY' \rangle & -\langle X^2 \rangle & \langle XX' \rangle \\
\langle XY' \rangle & 0 & \langle XX' \rangle & \langle X'^2 \rangle \\
\end{bmatrix}
\]

\[
\mathcal{L} = \frac{1}{2} \langle XY' \rangle - \langle XX' \rangle = \langle XY' \rangle - \langle YY' \rangle
\]

\[
\sqrt{\det[\Sigma]} = \langle X^2 \rangle \langle X'^2 \rangle - \langle XX' \rangle^2 - \langle XY' \rangle^2 = \epsilon_r^{2} - \mathcal{L}^2 = \epsilon_1 \epsilon_2
\]

\[
-\frac{1}{2} \text{tr}[(\Sigma J)^2] = 2 \left( \langle X^2 \rangle \langle X'^2 \rangle - \langle XX' \rangle^2 + \langle YY' \rangle^2 \right) = 2 \left( \epsilon_r^{2} + \mathcal{L}^2 \right) = \epsilon_1^2 + \epsilon_2^2
\]

* In this slide, we use the notation of $Z = (X, X', Y, Y')^T$ which is canonical in the Larmor frame
Round-to-flat transformation of angular-momentum-dominated beams

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(Received 13 June 2003; published 30 October 2003)

A study of round-to-flat configurations, and vice versa, of angular-momentum-dominated beams is presented. The beam propagation in an axial magnetic field is described in terms of the familiar Courant-Snyder formalism by using a rotating coordinate system. The discussion of the beam transformation is based on the general properties of a cylindrically symmetric beam matrix and the existence of two invariants for a symplectic transformation in 4D phase space.

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Experimental Demonstration

**Abstract**

Generation of angular-momentum-dominated electron beams from a photoinjector

Y.-E. Sun, P. Piot, K.-I. Kim, N. Barov, S. Lidia, J. Santucci, R. Tikhoplatov, and J. Wonnerberg

Various projects under study require an angular-momentum-dominated electron beam generated by a photoinjector. Some of the proposals directly use the angular-momentum-dominated beams (e.g., electron cooling of heavy ions), while others require the beam to be transformed into a flat beam (e.g., possible electron injectors for light sources and linear colliders). In this paper we report our experimental study of an angular-momentum-dominated beam produced in a photoinjector, addressing the dependencies of angular momentum on initial conditions. We also briefly discuss the removal of angular momentum. The results of the experiment, carried out at the Fermilab/NICADD Photoinjector Laboratory, are found to be in good agreement with theoretical and numerical models.

PACS numbers: 29.27.-a, 41.85.-p, 41.75.Fr

**References**

-Citations and further reading

**Conclusion**

The performance of accelerators benefits from phase-space tailoring by coupling of degrees of freedom. Previously applied techniques swap the emittances among the three degrees but the set of available emittances is fixed. In contrast to these emittance exchange scenarios, the emittance transfer scenario presented here allows for arbitrarily changing the set of emittances as long as the product of the emittances is preserved. This Letter is the first experimental demonstration of transverse emittance transfer along an ion beam line. The amount of transfer is chosen by setting just one single magnetic field value. The envelope functions (beta) and slopes (alpha) of the finally uncorrelated and repartitioned beam at the exit of the transfer line do not depend on the amount of transfer.

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• Beam matching in 2+ D
Applying steps in 1D matching

\[ \Sigma = \langle zz^T \rangle \rightarrow Q^{-1}P^{-1}\varepsilon P^{-T}Q^{-T}, \text{ with } \varepsilon = \tilde{\Sigma}(0) \]

A unitary matrix (symplectic rotation) in 2+ D

\[ P^{-1}\varepsilon P^{-T} = \varepsilon \rightarrow \text{This should be independent of particle’s phase advance} \]

Two possible cases:

\[ \varepsilon = \begin{bmatrix}
\varepsilon & 0 & 0 & 0 \\
0 & \varepsilon & 0 & 0 \\
0 & 0 & \varepsilon & 0 \\
0 & 0 & 0 & \varepsilon
\end{bmatrix}, \text{ or } \varepsilon = \begin{bmatrix}
\varepsilon_1 & 0 & 0 & 0 \\
0 & \varepsilon_2 & 0 & 0 \\
0 & 0 & \varepsilon_1 & 0 \\
0 & 0 & 0 & \varepsilon_2
\end{bmatrix} \]

\[ \theta : \text{defined in the next slide} \]

\[ Q(s) = Q(s + L) \]

But to meet the matching condition 2, \( \varepsilon \) should have a special form.

\( \theta \) : defined in the next slide

No motion across the eigen-planes

In principle, \( \varepsilon \) can be an arbitrary positive definite matrix
An arbitrary unitary matrix can be parametrized as (e.g., based on Sec. 3.3 of Sakurai):

\[ U(2) = e^{i\lambda} R(\alpha, \beta, \gamma) = e^{i\lambda} \exp(-i\sigma_3 \alpha/2) \exp(-i\sigma_2 \beta/2) \exp(-i\sigma_3 \gamma/2) = e^{i\lambda} \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \begin{pmatrix} \cos \beta/2 & -\sin \beta/2 \\ \sin \beta/2 & \cos \beta/2 \end{pmatrix} \begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix} \]

\[ \rightarrow \begin{pmatrix} \cos[\lambda] & 0 & -\sin[\lambda] & 0 \\ 0 & \cos[\lambda] & 0 & -\sin[\lambda] \\ \sin[\lambda] & 0 & \cos[\lambda] & 0 \\ 0 & \sin[\lambda] & 0 & \cos[\lambda] \end{pmatrix} \begin{pmatrix} \cos[(\xi + \eta)/2] & 0 & -\sin[(\xi + \eta)/2] & 0 \\ 0 & \cos[(\xi + \eta)/2] & 0 & -\sin[(\xi + \eta)/2] \\ \sin[(\xi + \eta)/2] & 0 & \cos[(\xi + \eta)/2] & 0 \\ 0 & \sin[(\xi + \eta)/2] & 0 & \cos[(\xi + \eta)/2] \end{pmatrix} = P \]

Here, \( \alpha = -(\xi + \eta), \beta/2 = \theta, \gamma = \eta - \xi \)
Two possible cases that make the above expression independent of the phase advance:

\[
P^{-1} \varepsilon P^{-T} = P^{-1} \begin{bmatrix}
\varepsilon_1 & 0 & 0 & 0 \\
0 & \varepsilon_2 & 0 & 0 \\
0 & 0 & \varepsilon_1 & 0 \\
0 & 0 & 0 & \varepsilon_2 \\
\end{bmatrix} P^{-T}
\]

\[
= \begin{pmatrix}
\cos[\theta]^2 \varepsilon_1 + \sin[\theta]^2 \varepsilon_2 & \frac{1}{2} \cos[\zeta - \eta] \sin[2\theta] (-\varepsilon_1 + \varepsilon_2) & \frac{1}{2} \sin[\zeta - \eta] \sin[2\theta] (\varepsilon_1 - \varepsilon_2) & \frac{1}{2} \sin[\zeta - \eta] \sin[2\theta] (-\varepsilon_1 + \varepsilon_2) \\
\frac{1}{2} \cos[\zeta - \eta] \sin[2\theta] (-\varepsilon_1 + \varepsilon_2) & \sin[\theta]^2 \varepsilon_1 + \cos[\theta]^2 \varepsilon_2 & 0 & \frac{1}{2} \sin[\zeta - \eta] \sin[2\theta] (\varepsilon_1 - \varepsilon_2) \\
0 & \frac{1}{2} \sin[\zeta - \eta] \sin[2\theta] (\varepsilon_1 - \varepsilon_2) & \cos[\theta]^2 \varepsilon_1 + \sin[\theta]^2 \varepsilon_2 & 0 \\
\frac{1}{2} \sin[\zeta - \eta] \sin[2\theta] (-\varepsilon_1 + \varepsilon_2) & 0 & \frac{1}{2} \cos[\zeta - \eta] \sin[2\theta] (-\varepsilon_1 + \varepsilon_2) & \sin[\theta]^2 \varepsilon_1 + \cos[\theta]^2 \varepsilon_2 \\
\end{pmatrix}
\]

\[
\rightarrow \begin{bmatrix}
\varepsilon_1 & 0 & 0 & 0 \\
0 & \varepsilon_2 & 0 & 0 \\
0 & 0 & \varepsilon_1 & 0 \\
0 & 0 & 0 & \varepsilon_2 \\
\end{bmatrix} \text{ with } \theta = 0
\]

\[
\rightarrow \begin{bmatrix}
(\varepsilon_1 + \varepsilon_2)/2 & 0 & 0 & 0 \\
0 & (\varepsilon_1 + \varepsilon_2)/2 & 0 & 0 \\
0 & 0 & (\varepsilon_1 + \varepsilon_2)/2 & 0 \\
0 & 0 & 0 & (\varepsilon_1 + \varepsilon_2)/2 \\
\end{bmatrix} \text{ with } \theta = \text{random}
\]
For special case:

If $\theta = 0$ (or, $\beta/2 = 0$)

$$U(2) = e^{i\lambda} \begin{pmatrix} e^{-i(\alpha+\gamma)/2} & 0 \\ 0 & e^{i(\alpha+\gamma)/2} \end{pmatrix}$$

Here, $\alpha = -(\xi + \eta), \gamma = \eta - \xi, \alpha + \gamma = -2\xi$

$$= \begin{pmatrix} e^{i(\lambda+\xi)} & 0 \\ 0 & e^{i(\lambda-\xi)} \end{pmatrix}$$

$$\mapsto \begin{pmatrix} \cos[\lambda + \xi] & 0 & -\sin[\lambda + \xi] & 0 \\ 0 & \cos[\lambda - \xi] & 0 & -\sin[\lambda - \xi] \\ \sin[\lambda + \xi] & 0 & \cos[\lambda + \xi] & 0 \\ 0 & \sin[\lambda - \xi] & 0 & \cos[\lambda - \xi] \end{pmatrix}$$

$\Rightarrow$ This is a typical form of the double rotation.
How to calculate Q?

• No universal standard:

• Solving matrix envelope equation:

\[ W = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix}, \quad W'' + \kappa W = (W^TWW^T)^{-1}, \quad Q = \begin{bmatrix} W^{-1} & 0 \\ -W^T \end{bmatrix} \]

Equivalence between various methods
On-going research
Methods for Linear Coupled Optics

1) **By decoupling transformation:** directly decouple the one-turn transfer map into an uncoupled one-turn map (i.e., into a block-diagonal form) through a matrix similarity transformation


2) **Using eigenvectors of the transfer matrix:** a transformation is found from the eigenvectors of the transfer matrix that puts the transfer matrix into “normal form”, i.e., the transfer matrix is transformed into a pure rotation

Conclusions

• Beam matching in 1 D $\rightarrow$ well-established, well-known

• Eigen-emittance $\rightarrow$ well-established, not well-known

• Beam matching in 2+ D $\rightarrow$ not completely established, not well-known
Thank you for your attention!