Lecture 3
‘Collective’ Descriptions of Beam Distributions

(Ch. 5 of FOBP)

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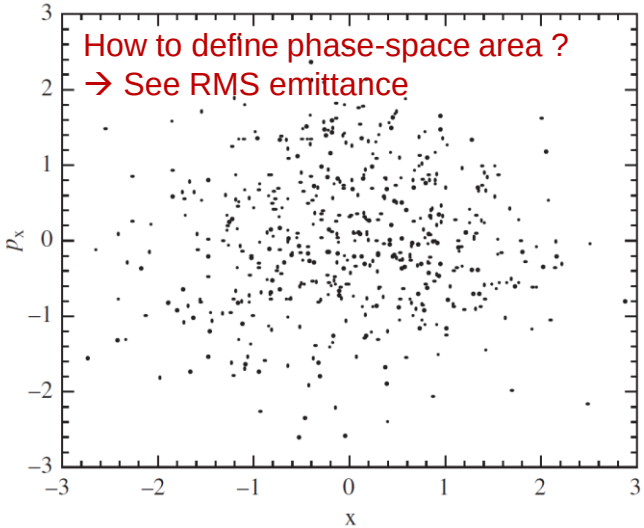
Distribution Function

- Distribution function:

Discrete (real world)



Continuous (mathematical approximation)

$$\longrightarrow f(\mathbf{x}, \mathbf{p}, t) d^3\mathbf{x} d^3\mathbf{p} =$$

The number of particles found in a differential volume in the neighborhood of a phase space location ***x, p*** at a time *t*

- With a smooth phase space distribution, the charge and current distributions associated with such a distribution are also continuous and smooth.

- The fields derived from the smooth charge/current densities may be termed macroscopic. Deviations from these approximate fields (near an individual particle) may be termed microscopic.

Liouville's Theorem

[lju'vil]

- Total time derivative of the distribution function:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\mathbf{x}} \cdot \nabla f + \dot{\mathbf{p}} \cdot \nabla_{\mathbf{p}} f$$

- From continuity in phase-space:

$$0 = \frac{\partial f}{\partial t} + \nabla \cdot (\dot{\mathbf{x}} f) + \nabla_{\mathbf{p}} \cdot (\dot{\mathbf{p}} f)$$

- If the forces are derivable from a Hamiltonian:

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial t} + \sum_i \left(\frac{dx_i}{dt} \frac{\partial f}{\partial x_i} + \frac{dp_i}{dt} \frac{\partial f}{\partial p_i} \right) \\ &= -\sum_i \left(\frac{\partial \dot{x}_i}{\partial x_i} f + \frac{\partial \dot{p}_i}{\partial p_i} f \right) = -\sum_i f \left[\frac{\partial}{\partial x_i} \left(\frac{\partial H}{\partial p_i} \right) - \frac{\partial}{\partial p_i} \left(\frac{\partial H}{\partial x_i} \right) \right] = 0 \end{aligned}$$

- In other words, when no dissipative forces, no particle lost or created, and no small-impact-parameter binary Coulomb collisions between particles:

$$\frac{df}{dt} = 0$$

Continuity: $0 = \frac{\partial f}{\partial t} + \nabla \cdot (\dot{\mathbf{x}} f) + \nabla_{\mathbf{p}} \cdot (\dot{\mathbf{p}} f) \longrightarrow \nabla \cdot (\dot{\mathbf{x}}) + \nabla_{\mathbf{p}} \cdot (\dot{\mathbf{p}}) = 0$:Imcompressibility

Some Comments on Liouville's Theorem

- Liouville's theorem states that the phase space density encountered as one travels with a particle in a Hamiltonian system is conserved.
  - The density of any volume of phase space whose boundary follows the Hamiltonian equations is constant.
  - The volume occupied by particles in phase space (~emittance) is conserved (shape may change).

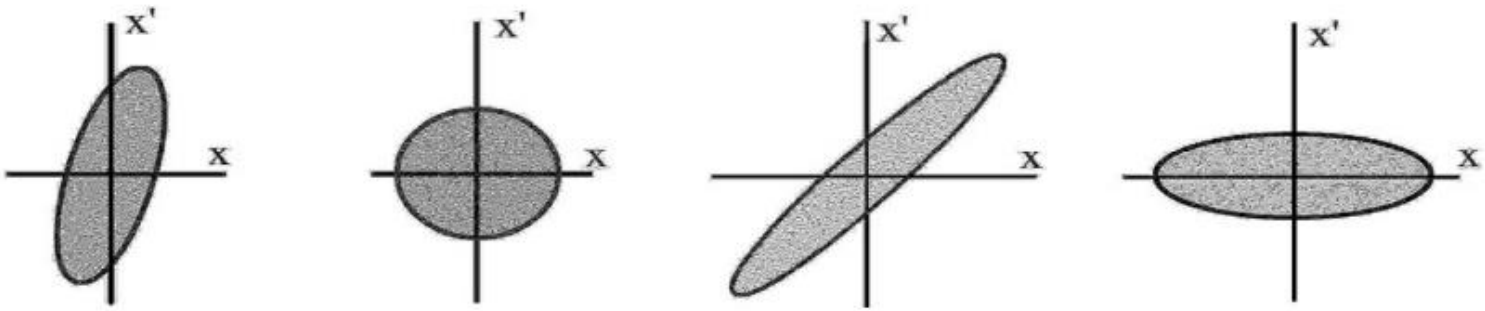
- Liouville's theorem is valid not only for the time-independent Hamiltonian case, but also for the time-dependent Hamiltonian case.
- Liouville's theorem is valid for both equilibrium and non-equilibrium systems.
- Liouville's theorem is valid for both linear and non-linear systems.

- Liouville's theorem does not imply that the density is uniform throughout phase space.
- Liouville's theorem only holds in the limit that the particles are infinitely close together. Equivalently, Liouville's theorem does not hold for any ensemble that consists of a finite number of particles.
- Liouville's theorem holds even in the presence of space-charge and wake-fields, but not with microscopic binary collisions.

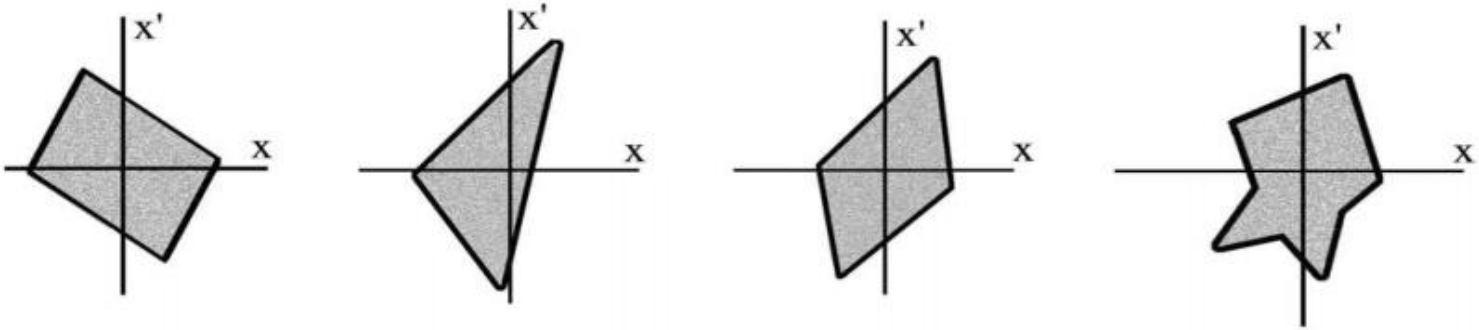
Emittance

(x, x') (x, x', y, y')

The phase space area/volume (~emittance) occupied by a particle beam is an invariant. (Incompressible flow by definition)



Phase space 'footprints' of the same beam



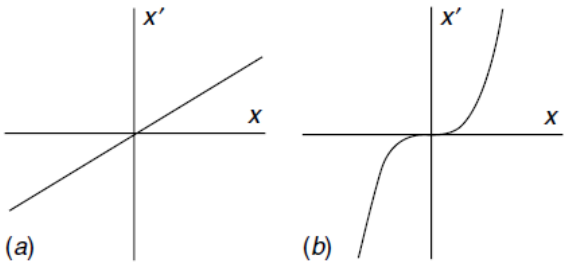
RMS Emittance

- In the case of a real beam with a finite number of particles (N), an RMS emittance can be defined for an effective phase-space (or trace-space) area (or volume).

$$\epsilon_{\rm rms} = \sqrt{\langle x^2\rangle \langle p_x^2\rangle - \langle x p_x\rangle^2}, \text{ or } \sqrt{\langle x^2\rangle \langle x'^2\rangle - \langle x x'\rangle^2}$$

→ Depends not only on the true area occupied by the beam in phase space, but also on the distortions produced by nonlinear forces.

→ Hamiltonian flow: Phase-space conservation + Entropy growth: Filamentation



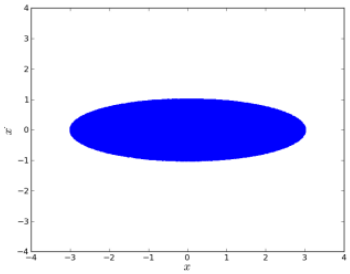
Phase-space area = 0 RMS emittance = 0

Phase-space area = 0 RMS emittance > 0

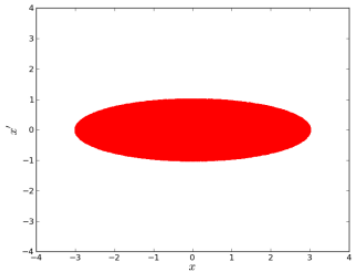
- However, when nonlinear forces act on the system, e.g. nonlinear magnetic fields, space charge force, the RMS emittance is not conserved.

Filamentation
→ Dilution of phase space density

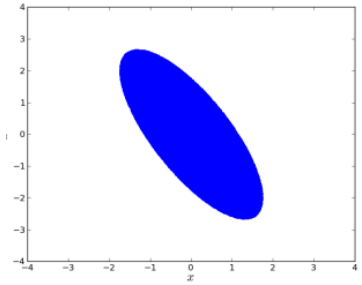
Linear force



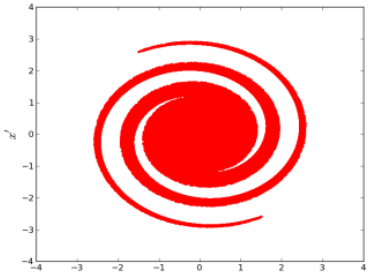
Nonlinear force



Linear force



Nonlinear force



For finite N, average distance of the particles in one spiral equals the distance between two adjacent spirals.

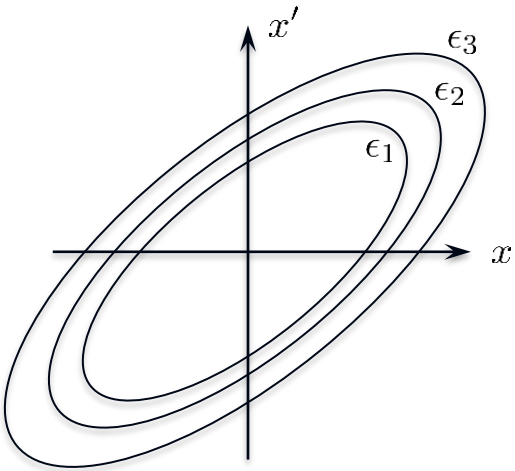
Secs. 5.2/5.3/5.4 of FOBP

Beam Distribution and Emittance

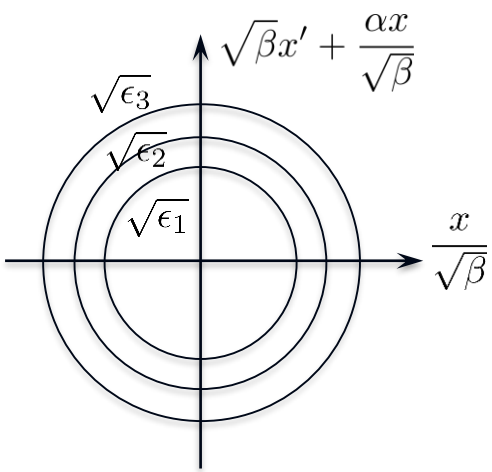
Bi-Gaussian distribution

- We assume the particle distribution is a bi-Gaussian distribution in the following form:

$$f(x,x') = \frac{1}{2\pi\epsilon_{\rm rms}} \exp\left[-\frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{2\epsilon_{\rm rms}}\right] \propto \exp\left[-\frac{\epsilon}{2\epsilon_{\rm rms}}\right] \propto \exp\left[-\frac{(x/\sqrt{\beta})^2 + (\sqrt{\beta}x' + \alpha x/\sqrt{\beta})^2}{2\epsilon_{\rm rms}}\right]$$



Constant (single particle) emittance ellipses define contours of constant phase-space distribution density



Constant (single particle) emittance circles in the normalized coordinates define contours of constant phase-space distribution density

- The rms beam emittance is proportional to the average of all the single particle emittances.
- The rms beam emittance is defined through the ellipse of the exp[-1/2] contour relative to the peak density contour.

Normalization of the distribution function

- First, check the normalization:

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,x')dxdx' &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\epsilon_{\rm rms}} \exp\left[-\frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{2\epsilon_{\rm rms}}\right] dxdx' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\epsilon_{\rm rms}} \exp\left[-\frac{x_n^2 + x_n'^2}{2\epsilon_{\rm rms}}\right] dx_n dx_n' \\
&= \int_{0}^{\infty} \frac{1}{2\pi\epsilon_{\rm rms}} \exp\left[-\frac{\epsilon}{2\epsilon_{\rm rms}}\right] \pi d\epsilon \\
&= 1
\end{aligned}
$$

$$x_n = \frac{x}{\sqrt{\beta}}$$

$$x_n' = \sqrt{\beta}x' + \frac{\alpha x}{\sqrt{\beta}}$$

$$\epsilon = x_n^2 + x_n'^2$$

- Meaning of the rms beam emittance:

Integration by parts

$$
\begin{aligned}
\langle\epsilon\rangle &= \int_0^{\infty} \epsilon \frac{1}{2\epsilon_{\rm rms}} \exp\left[-\frac{\epsilon}{2\epsilon_{\rm rms}}\right] d\epsilon \\
&= \frac{1}{2\epsilon_{\rm rms}} \left\{ \epsilon(-2\epsilon_{\rm rms}) \exp\left[-\frac{\epsilon}{2\epsilon_{\rm rms}}\right]\Bigg|_0^{\infty} + \int_0^{\infty} 2\epsilon_{\rm rms} \exp\left[-\frac{\epsilon}{2\epsilon_{\rm rms}}\right] d\epsilon \right\} \\
&= 2\epsilon_{\rm rms}
\end{aligned}
$$

Moments of the distribution function

- From the general properties of the bi-Gaussian distribution in (x, y) plane:

https://en.wikipedia.org/wiki/Multivariate_normal_distribution

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y(1-\rho^2)^{1/2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{\delta x^2}{\sigma_x^2} - 2\rho\frac{\delta x\delta_y}{\sigma_x\sigma_y} + \frac{\delta y^2}{\sigma_y^2}\right)\right]$$

Where

$$\delta x = x - \langle x \rangle\,, \quad \delta y = y - \langle y \rangle$$

$$\sigma_x^2 = \left\langle \delta x^2 \right\rangle, \quad \sigma_y^2 = \left\langle \delta y^2 \right\rangle, \quad \sigma_{xy} = \langle \delta x \delta y \rangle \equiv \rho\sigma_x\sigma_y$$

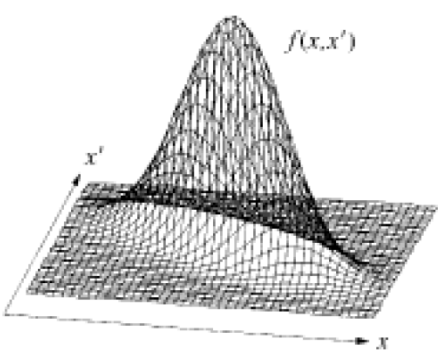
covariance

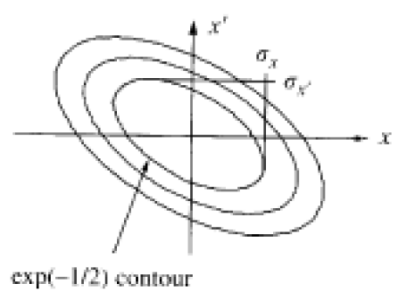
- By comparing with the beam distribution in (x, x') space:

$$\langle x \rangle = \langle x' \rangle = 0 \;\; \text{when beam is alinged to its design axis}$$

$$\sigma_x^2 = \left\langle x^2 \right\rangle = \epsilon_{\rm rms}\beta, \quad \sigma_{x'}^2 = \left\langle x'^2 \right\rangle = \epsilon_{\rm rms}\gamma, \quad \sigma_{xx'} = \langle xx' \rangle = -\epsilon_{\rm rms}\alpha$$

$$\epsilon_{\rm rms} = \sqrt{\sigma_x^2\sigma_{x'}^2 - \rho^2\sigma_x^2\sigma_{x'}^2} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$





$\pi\epsilon_{\rm rms}$ = Area of the exp[-1/2] contour

Beam matrix

- The beam matrix is the second-order moments of the beam distribution:

$$
\begin{aligned}
\sigma(s) = \mathbf{\Sigma}(s) &= \left\langle \mathbf{x}\mathbf{x}^T \right\rangle \\
&= \begin{bmatrix} \left\langle x^2 \right\rangle & \left\langle xx' \right\rangle \\ \left\langle xx' \right\rangle & \left\langle x'^2 \right\rangle \end{bmatrix} = \begin{bmatrix} \sigma_x^2 & \sigma_{xx'} \\ \sigma_{xx'} & \sigma_x'^2 \end{bmatrix} \\
&= \underbrace{\epsilon_{\mathrm{rms}}}_{\text{Beam property}} \underbrace{\begin{bmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{bmatrix}}_{\text{Lattice properties}}
\end{aligned}
$$

Contains all the necessary information describing the beam

If the beam aligns with Courant-Snyder parameters

- Note that the determinant of the beam matrix is the rms emittance:

$$\det(\sigma) = \left\langle x^2 \right\rangle \left\langle x'^2 \right\rangle - \left\langle xx' \right\rangle^2 = \epsilon_{\mathrm{rms}}^2$$

- If the transfer matrix is known,

$$\mathbf{x}(s) = \mathbf{M}_{s_0 \to s} \cdot \mathbf{x}(s_0)$$

$$
\begin{aligned}
\sigma(s) &= \left\langle \mathbf{x}(s)\mathbf{x}^T(s) \right\rangle \\
&= \left\langle \mathbf{M}_{s_0 \to s} \cdot \mathbf{x}(s_0)\mathbf{x}^T(s_0) \cdot \mathbf{M}_{s_0 \to s}^T \right\rangle \\
&= \mathbf{M}_{s_0 \to s} \cdot \sigma(s_0) \cdot \mathbf{M}_{s_0 \to s}^T
\end{aligned}
$$

Fraction of particles enclosed

- From the normalization of the distribution function in slide 9:

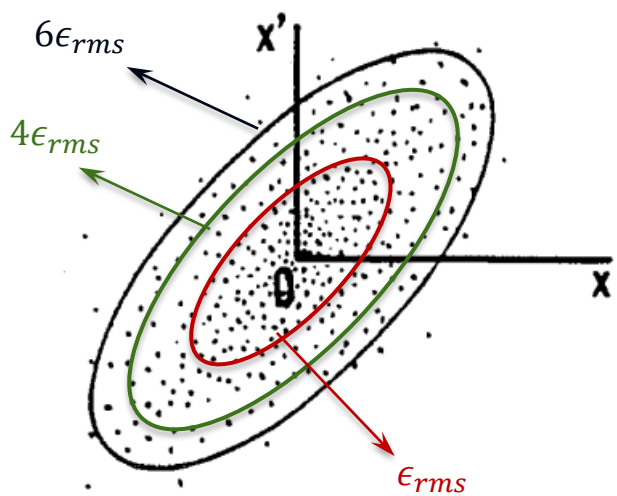
$$F = \int_0^{\epsilon_F} \frac{1}{2\epsilon_{\rm rms}} \exp\left[-\frac{\epsilon}{2\epsilon_{\rm rms}}\right] d\epsilon$$

  - Note that if $\epsilon_F \to \infty, F = 100\%$.
  - The ϵ_F indicates the emittance value with encloses $F(\%)$ fraction of the particles.

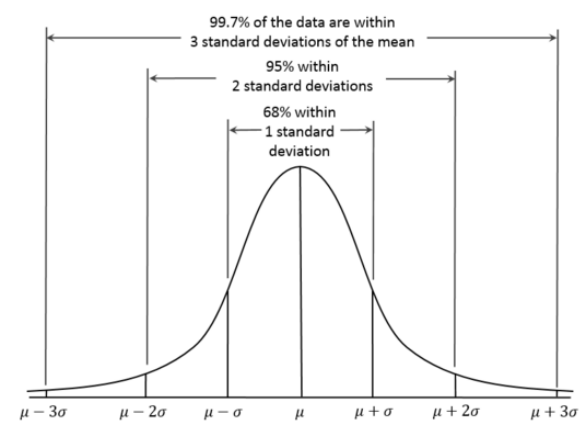
$$F = -\exp\left[-\frac{\epsilon}{2\epsilon_{\rm rms}}\right]\Bigg|_0^{\epsilon_F} = 1 - \exp\left[-\frac{\epsilon_F}{2\epsilon_{\rm rms}}\right]$$

$$\epsilon_F = -2\epsilon_{\rm rms}\ln(1-F)$$

ϵ_F	$F(\%)$
0	0
ϵ_{rms}	39%
$4\epsilon_{rms}$	87%
$6\epsilon_{rms}$	95%
∞	100%



Be careful! It is different from the single Gaussian



If the beam is not in thermal equilibrium:

- We used bi-Gaussian distribution assuming that the beam is in thermal (stationary) equilibrium:

$$\frac{\partial f}{\partial t} = 0, \quad f \propto \exp\left[-\frac{H}{k_B T}\right]$$

- Even though the beam distribution function is not exactly in thermal equilibrium, it is often used as a good approximation.

- For example, in the periodic focusing system, the particle motion is always non-equilibrium ($\partial f/\partial t \neq 0$, but $df/dt = 0$) however, when plotted in trace space once per period (i.e., in the Poincare plot), we can treat the beam in equilibrium.

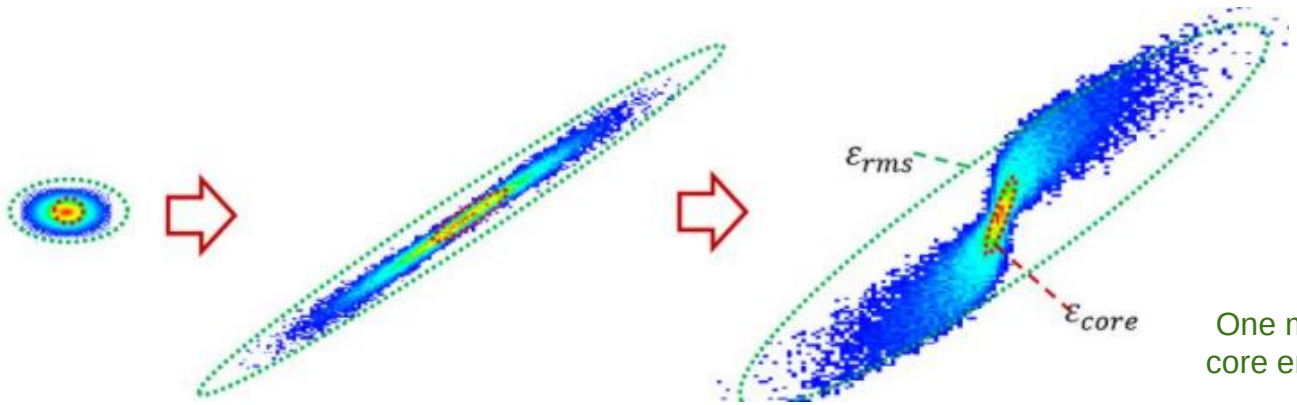
$$f(s) = f(s + L_p)$$

- Thermalization is often achieved very slowly, over many revolutions of a circular accelerator, by a combination of damping and heating effects (e.g., radiation emission, intra-beam scattering).

- In fast, transient systems, such as linear accelerators, equilibrating mechanisms (i.e. collisions) are too slow to be relevant, and if equilibria are found, they must be a property of the particle source used (Collective effects may enhance relaxation rate though).

If the focusing force is not linear:

- Due to the non-linear forces, which are not included in the Courant-Snyder model, beam trajectories may not be simply ellipses.



One may use core emittance

- Non-linear forces are induced by nonlinear magnetic fields and space charge forces, and increase the rms emittance → Still we can calculate the rms emittance and 2nd moments!

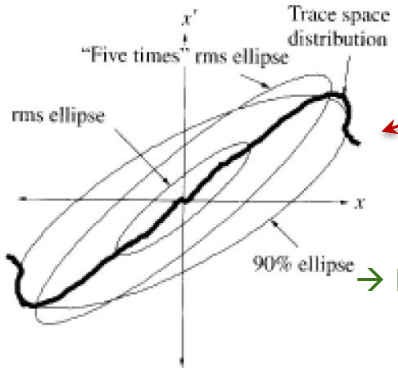
- The rms emittance depends not only on the true area occupied by the beam in phase space (which is constant by Liouville theorem), but also on the distortions produced by nonlinear forces (see slide 6).

If the beam is not matching with the ellipse:

- Strictly speaking, beam's elliptical shape and orientation determined by the second-moments may not match with the ellipse specified by the periodic lattice system:

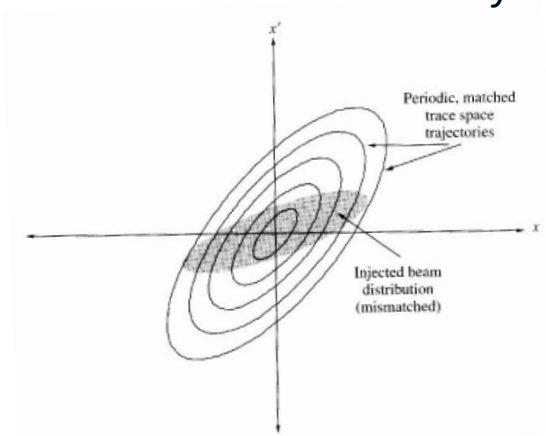
$$\beta_{beam} = \langle x^2 \rangle / \epsilon_{\rm rms} \neq \beta_{lattice}, \quad \gamma_{beam} = \langle x'^2 \rangle / \epsilon_{\rm rms} \neq \gamma_{lattice}, \quad \alpha_{beam} = -\langle x x' \rangle / \epsilon_{\rm rms} \neq \alpha_{lattice}$$

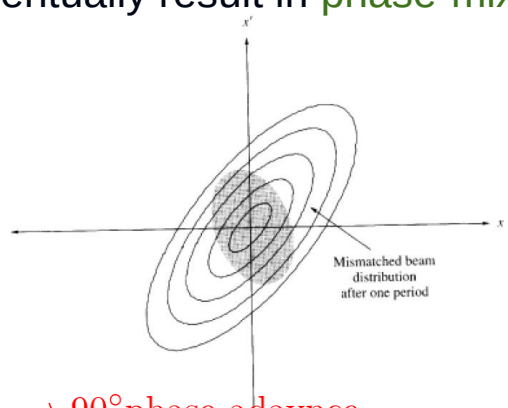
- Often, even beam's elliptical shape and orientation may not be unique. The second-moment definition of Twiss parameters can be anomalously dependent on "tail particles".



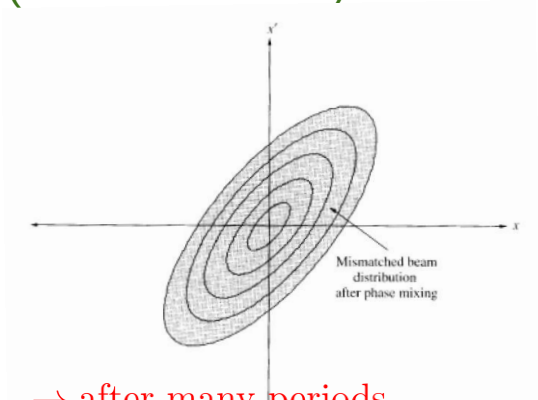
→ Minimum area ellipse containing 90% of the particles

- The mismatch may seem harmless at first glance. However, amplitude-dependent tune due to small nonlinearity will eventually result in phase-mixing (or de-coherence).





→ 90° phase adavnce



→ after many periods

Secs. 5.5 of FOBP

The RMS Envelope Equation and Normalized Emittance

The rms envelope equation

- The first derivative of the rms beam size:

$$
\begin{aligned}
\frac{d\sigma_x}{ds} &= \frac{d}{ds}\sqrt{\langle x^2\rangle} = \frac{1}{2\sigma_x}\frac{d}{ds}\langle x^2\rangle \\
&= \frac{1}{2\sigma_x}\frac{d}{ds}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_x(x,x')dxdx' \\
&= \frac{1}{\sigma_x}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xx' f_x(x,x')dxdx' \\
&= \frac{\langle xx'\rangle}{\sigma_x} = \frac{\sigma_{xx'}}{\sigma_x}
\end{aligned}
$$

- The second derivative of the rms beam size:

$$
\begin{aligned}
\frac{d^2\sigma_x}{ds^2} &= \frac{d}{ds}\frac{\sigma_{xx'}}{\sigma_x} = \frac{1}{\sigma_x}\frac{d\sigma_{xx'}}{ds} - \frac{\sigma_{xx'}^2}{\sigma_x^3} \\
&= \frac{1}{\sigma_x}\frac{d}{ds}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xx' f_x(x,x')dxdx' - \frac{\sigma_{xx'}^2}{\sigma_x^3} \\
&= \frac{\sigma_{x'}^2 + \langle xx''\rangle}{\sigma_x} - \frac{\sigma_{xx'}^2}{\sigma_x^3} \\
&= \frac{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2}{\sigma_x^3} - \frac{\langle xx''\rangle}{\sigma_x} \\
&= \frac{\epsilon_{\mathrm{rms}}^2}{\sigma_x^3} - \frac{\langle xx''\rangle}{\sigma_x}
\end{aligned}
$$

$$\epsilon_{\mathrm{rms}} = \sqrt{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2}$$

The rms envelope equation (cont'd)

- For the linear transport conditions,

$$x'' + \kappa_x^2 x = 0 \longrightarrow \langle x x'' \rangle = -\kappa_x^2 \langle x^2 \rangle = -\kappa_x^2 \sigma_x^2$$

- Thus

$$\sigma_x'' + \kappa_x^2 \sigma_x = \frac{\epsilon_{\mathrm{rms}}^2}{\sigma_x^3}$$

  - The envelope evolution is controlled by the above equation.
  - The homogeneous portion of the equation is identical to that of a single particle.
  - The inhomogeneous term on the right hand side can be interpreted mathematically as the outward forcing of the beam envelope by the rms spread in trajectory angle, which is parametrized by the non-vanishing rms emittance.
  - It can also be interpreted physically in terms of the outward pressure in the beam region due to the thermal nature of the collection of beam particles.

The effect of acceleration

- In the paraxial approximation,

$$x' = \frac{dx}{ds} = \frac{v_x}{v_z} \simeq \frac{p_x}{p_0}, \quad p_0 = \beta_0\gamma_0 m_0 c$$

$$x'' = \frac{p_x'}{p_0} - \frac{p_0'}{p_0}x' = \frac{p_x'}{p_0} - \frac{(\beta_0\gamma_0)'}{(\beta_0\gamma_0)}x'$$

- Since momentum is used, we recall the equation of motion

$$F_x = -v_0 q B' x = -p_0 v_0 \kappa_x^2 x = \frac{dp_x}{dt} \longrightarrow \frac{p_x'}{p_0} = -\kappa_x^2 x$$

$$\frac{d^2\sigma_x}{ds^2} = \frac{d}{ds}\frac{\sigma_{xx'}}{\sigma_x} = \frac{1}{\sigma_x}\frac{d\sigma_{xx'}}{ds} - \frac{\sigma_{xx'}^2}{\sigma_x^3}$$

$$= \frac{1}{\sigma_x}\left[\sigma_{x'}^2 - \kappa_x^2\sigma_x^2 - \frac{(\beta_0\gamma_0)'}{(\beta_0\gamma_0)}\sigma_{xx'}\right] - \frac{\sigma_{xx'}^2}{\sigma_x^3}$$

$$\frac{d}{ds}\langle xx'\rangle = \langle x'^2\rangle + \langle xx''\rangle$$

- In more standard form:

$$\frac{d^2\sigma_x}{ds^2} + \underline{\frac{(\beta_0\gamma_0)'}{(\beta_0\gamma_0)}\frac{d\sigma_x}{ds}} + \kappa_x^2\sigma_x = \frac{\epsilon_{\mathrm{rms}}^2}{\sigma_x^3} = \frac{\epsilon_n^2}{(\beta_0\gamma_0)^2\sigma_x^3}$$

Normalized emittance

- We introduced the normalized emittance:

$$\epsilon_n = \beta_0 \gamma_0 \epsilon_{\mathrm{rms}}$$

$$\epsilon_n^2 = (\beta_0\gamma_0)^2 \left[\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right] = (m_0 c)^{-2} \left[\langle x^2\rangle\langle p_x^2\rangle - \langle xp_x\rangle^2\right]$$

- The normalized emittance (not the rms emittance in trace space) is, in fact, invariant under combined effects of linear transverse forces and longitudinal acceleration.

- This result is a direct consequence of the adibatic damping of beam particle angle under acceleration, which causes the emittance defined in trace space to be diminished.

- The invariant normalized emittance is an effective area occupied by the beam in the phase plane, not the trace plane.